THE THEORY

OF

MODERN OPTICAL INSTRUMENTS

A REFERENCE BOOK FOR PHYSICISTS
MANUFACTURERS OF OPTICAL INSTRUMENTS
AND FOR OFFICERS IN THE ARMY AND NAVY

BY

DR. ALEXANDER GLEICHEN

TRANSLATED FROM THE GERMAN BY

H. H. EMSLEY, B.Sc., AND W. SWAINE, B.Sc.

SECOND EDITION

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PART I.

GENERAL THEORY.

CHAPTER I.

Fundamental Ideas. The Laws of Reflection and Refraction. Dispersion. Prisms.

1. Fundamental Principles.

When a light disturbance emanates from a point in a transparent isotropic medium, as, for example, optical glass and many liquids, it spreads out in all directions with a *constant* velocity. In general this velocity varies in different media. Calling the velocity in any given medium v, and the velocity in vacuo v_0, then

$$n = \frac{v_0}{v}$$

is the **refractive index** of the medium. For a second medium of refractive index n' and in which the velocity of propagation of the disturbance is v', then

$$n' = \frac{v_0}{v'}$$

From these two equations we have

$$\frac{n}{n'} = \frac{v'}{v}$$

The refractive indices of two media are inversely proportional to the velocities of light in those media. If the origin of the disturbance be very small, it may be called **a luminous point.** Straight lines drawn from this point in the direction of propagation of the light disturbance are

called **rays of light.** Along these rays the light radiates in the form of waves, whilst every small particle of the medium executes very small vibrations perpendicular to the direction of propagation. If τ is the period of vibration, λ the wave-length, we have the fundamental law of Physical Optics:

$$\lambda = v\tau$$

At any given instant the disturbance reaches a surface perpendicular to the rays of light called a **wave surface,** which, in the case of an isotropic medium, is a sphere. Every point in which the rays of light, produced if necessary, intersect, is called as a generalisation a luminous point. The sum total of such rays running infinitely near to one another is called an infinitely thin **bundle** of rays. If the rays occupy a finite portion of space we may speak simply of a bundle of rays.

If a bundle of rays strike the surface of separation of two media, its subsequent behaviour depends on the nature of this surface. If the second medium be opaque and the surface rough, the rays will be thrown back in all directions (*irregular* or *diffuse* reflection); if the surface be smooth, the rays will be thrown back or reflected according to the **law of reflection** (*regular* reflection). If this second medium be also transparent, then part of the light will be regularly reflected, and the remainder will penetrate the second medium according to the **law of refraction.**

2. Law of Reflection.

The incident and reflected rays lie in the same plane as the normal to the reflecting surface at the point of incidence, and make equal angles with the normal on opposite sides.

In Fig. 1, LL_1 is the trace of the reflecting surface in the plane of incidence, which coincides with the plane of the paper. A ray from P strikes the surface at A. MAN is the normal at A. The reflected ray is AP'. Then according to the law of reflection:

$$P\hat{A}N = P'\hat{A}N$$

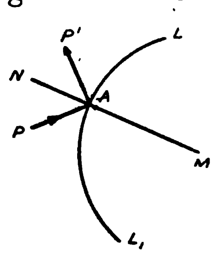

FIG. 1.

From the law of reflection* it follows: All rays from a luminous point P incident on a plane mirror are reflected

* For a fuller statement of the law of reflection *see* the Author's "Lehrbuch der geometrischen Optik." Leipzig, 1902, pp. 46–66.

as if they came from a point P', which lies as far behind the mirror as P is in front of it. We call P' the **image** corresponding to the point P; and since this point P' does not actually exist, it is called **virtual**, in contradistinction to a **real** image which actually exists in space at the point of intersection of the rays.

3. Law of Refraction.*

The incident and refracted rays are in the same plane with the normal at the point of incidence, and on opposite sides of it. The ratio of the sine of the angle of incidence to the sine of the angle of refraction is a constant. If n is the refractive index of the first medium and n' that of the second medium, then:

$$n \sin i = n' \sin i' \qquad \ldots \qquad \ldots \qquad (1)$$

In Fig. 2, LL_1 is the trace of the refracting surface in the plane of incidence, which coincides with the plane of the paper. The incident ray is PA. The refracted ray is AP' and the normal to the surface is NAM, which also lies in the plane of incidence.

$$P\hat{A}N = i \qquad P'\hat{A}M = i'$$

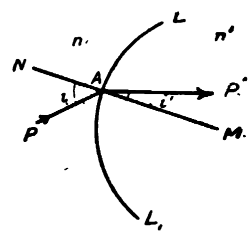

FIG. 2.

The angle of incidence i and the angle of refraction i' lie on opposite sides of the normal.

If $n' > n$, then $i' < i$; if $n' < n$, then $i' > i$.

4. Total Reflection.

If in the equation $n \sin i = n' \sin i'$, n be greater than n', it may so happen that for a certain angle of incidence i there can be no angle of refraction i'; for since the sine of an angle cannot be greater than unity, the greatest angle which the refracted ray may make with the incidence normal is determined by $i' = \sin^{-1} 1 = 90°$. In this case the law of refraction becomes

$$\sin i = \frac{n'}{n}$$

This angle i is called the **critical angle** of total reflection. For example, if a ray traversing glass of refractive index $n = \frac{3}{2}$, is incident on a plane surface separating the glass from air ($n' = 1$), then we have

$$\sin i = \tfrac{2}{3},$$

* Sometimes called Snell's Law from its discoverer Willebrord Snell, of Leyden; 1626.—Trans.

giving $i = 41° 49'$. The refracted ray corresponding to this incident ray travels *along* the surface of separation, and is known as the **grazing** ray. Rays traversing the glass and falling more obliquely on the surface are reflected back into the glass; these are called **totally reflected** rays.

In Fig. 3, P is a point in glass; EE_1 is the surface of separation between the glass below and air above. The ray PA_1 is refracted into the air; the ray PA_2, for which $i = 41° 49'$ is the critical ray, the corresponding refracted ray A_2E_1 grazing the surface; the ray PA_3, for which $i > 41° 49'$, is totally reflected.

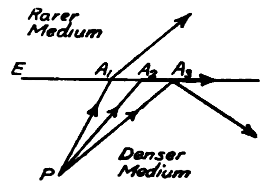

FIG. 3.

5. Refraction of Light through a Prism.

In Geometrical Optics, a prism is defined as a refracting medium bounded by intersecting plane surfaces. In particular, it is a medium bounded by *two* planes. The straight line in which these planes intersect is called the **Edge** of the prism; the included angle is called the **Refracting Angle**, and a section perpendicular to the edge a **Principal Section.** An incident ray, travelling in the principal section remains, after refraction, in the same plane, because the perpendicular erected at the point of incidence lies in the principal section. Fig. 4 represents a principal section of a prism of refractive index n, surrounded by air, the refractive index of which we may take to be equal to unity.*

Let S be the point in which the edge cuts the principal section (plane of paper). The ray $PABP'$ from P, traversing the principal section, makes at the point of incidence A on the first surface the angles i and i' with the perpendicular at A; at B, the ray makes with the incidence normal the angles i_1 and i_1'. The incidence normals intersect in C. We will represent the refracting angle ASB by σ, and the angle at D, which the emerging ray makes with the original incident ray, the deviation ω; then from Fig. 4,

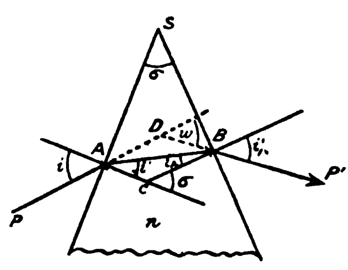

FIG. 4.

$$\sigma = i' + i_1 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$
$$\omega = i + i_1' - \sigma \quad \ldots \quad \ldots \quad (3)$$

* The refractive index of air is 1·00029.

Further, it follows from the Law of Refraction :

$$\sin i = n \sin i'' \qquad \ldots \qquad \ldots \qquad (4)$$
$$n \sin i_1 = \sin i_1' \qquad \ldots \qquad \ldots \qquad (4a)$$

If a ray traverse the prism symmetrically, so that $i = i_1'$ and $i'' = i_1$, then it can be shown that the deviation ω will be a minimum.*

If the angles of incidence and refraction be very small, their circular measures may be substituted for their sines, and the last two equations become :

$$i = n\, i''$$
$$n\, i_1 = i_1'$$

and introducing these into equations (2) and (3) we obtain†

$$\omega = (n-1)\,\sigma \qquad \ldots \qquad \ldots \qquad (5)$$

Ex. 1.

In a prism of refracting angle $\sigma = 3^c$, and refractive index $n = 1{\cdot}62$, what is the total deviation it produces?

From equation (5)

$$\omega = 0{\cdot}62 \times 3^\circ = 1^\circ{\cdot}86 = 1^\circ\, 52'.$$

If a ray be caused to pass through several prisms, the total deviation produced is equal to the sum of the deviations produced by the separate prisms. If the prisms be placed with their refracting angles opposed, the deviations are to be reckoned with opposite signs.

6. Plates with Parallel Faces.

If the refracting angle σ be equal to zero, we have a parallel-sided plate. Then equation (2) becomes $i'' = -i_1$; (4) and (4*a*) give $i = -i_1'$; hence $\omega = 0$ from (3).

A parallel-sided plate does not change the direction of a ray passing through it, but, if the incidence be oblique, merely gives it a lateral displacement.

7. Dispersion of Light.

The refractive index of a medium varies with the colour of the light.

White light, as we learn from physical optics, is a combination of light of different colours, which have different refractive indices in any given medium. By refraction, a beam of white light is decomposed into a number of

---

* *See* Gleichen "Lehrbuch der geometrischen Optik." Leipzig, 1902, p. 33.

† In a prism of small refracting angle, for small angles of incidence the deviation is independent of the angle of incidence. This is of importance in the Rangefinder.—Trans.

coloured beams. This phenomenon is called **Dispersion.** If we use a prism as a refracting medium, and allow a narrow beam of white light to fall on it through a slit parallel to the edge of the prism, then this white beam is decomposed by the prism into a fan of coloured beams, all still parallel to the edge of the prism. Of these colours, the eye distinguishes in particular the following, corresponding to an ascending order of refractive indices: red, orange, yellow, green, blue, indigo, violet. We may receive this fan of colours on a white screen, or observe it by means of an optical instrument, and see a band of colours, the so-called **Spectrum,** with the colours in the above order, and parallel to the slit. The spectrum of the light from the sun is crossed by a considerable number of fine dark lines parallel to the slit; and no matter by what refracting medium the spectrum is produced, these lines always appear in the same colours, and consequently are especially adapted to the precise definition of those colours (Fraunhofer Lines).* The light from incandescent solid bodies yields continuous spectra, whilst the light from incandescent vapours yields a number of differently coloured lines, separated by dark intervals. The most important Fraunhofer lines are designated by the letters A to H, distributed over the visible spectrum from the red end to the violet.

Many new kinds of glass with widely varying properties have been produced in recent years, and their dispersions are of particular importance in the theory of optical instruments. In their glass works at Jena, Schott & Co. make use of the following lines, produced by incandescent vapours, for the determination of the refractive indices of the various kinds of glass they manufacture. By using these artificial sources of light, sunlight can be dispensed with in the determination of the data.†

1. The red potassium line A', which falls very close to the Fraunhofer line A.
2. The yellow double line of sodium, which coincides with the Fraunhofer line D.
3. The three bright hydrogen lines, of which the first two are identical with the Fraunhofer lines C and F; and the third of which, called G', lies very near to the Fraunhofer line G.

* So called after their discoverer, Joseph Fraunhofer, of Munich.—Trans.

† For a more detailed account of this subject *see* Hovestadt "Jena Glass and its Applications." Macmillan, 1902.—Trans.

The spectra produced by prisms with equal refracting angles, but of different kinds of glass, are, in general, neither coincident nor similar.

By the **mean refractive index** of a medium is understood the refractive index of the medium for the light corresponding to the D lines ; in what follows, this is what is meant by the term refractive index, unless otherwise stated.

By the **mean dispersion** is understood the difference between the refractive indices of the medium for the light corresponding to the lines F and C ; *i.e.*

$$\textbf{mean dispersion} = dn = n_F - n_C$$

Glasses with comparatively small dn and small n_D are called *Crown* glasses ; and those with large n_D and large dn, *Flint* glasses. There is not, however, any strict line of demarcation between the various kinds of glass, as in recent years glasses have been produced with comparatively large n_D and small dn (*e.g.*, Barium crown).

Schott and Co. in their catalogues, classify glasses according to the term

$$\nu = \frac{n_D - 1}{n_F - n_C}$$

ν being the reciprocal of the **Dispersive Power.** Their lists also give the values of n_D, and of the so-called **relative partial dispersions**

$$\frac{n_D - n_{A'}}{n_F - n_C}; \quad \frac{n_F - n_D}{n_F - n_C}, \quad \frac{n_{G'} - n_F}{n_F - n_C}$$

on which quantities depend the **Secondary spectra.***

---

* In Fraunhofer's time and prior to 1886, crown and flint glasses were produced with restricted optical properties. Crown glasses had lower refractive index and smaller dispersion, flints had higher index and higher dispersion. The production of a wider diversity of glasses by Messrs. Schott & Co., following on the pioneer work of Vernon Harcourt, marked a pronounced epoch in the manufacture of optical instruments. The older silicate glasses were augmented by phosphate and borate glasses, whose special utility lay in their eliminating the secondary spectrum. The ν-value, which is equal to $\frac{n_D - 1}{n_F - n_C}$, describes the relative refractivity of a glass for equal mean dispersion. The relative partial dispersions describe the relative dispersions in different parts of the spectrum for equal mean dispersion. Equality of these latter ratios, therefore, expresses the similarity of the spectra produced by various glasses. By means of the phosphate and borate glasses, pairs of crown and flint glass may be chosen with similar "run of dispersion" or equal relative partial dispessions.

Further barium glasses were produced in which low dispersive power was associated with high refractivity, to which reference will be made in the discussion on the construction of achromatic lenses and the photographic objective.—Trans.

See Gleichen "Lehrbuch der geom. Optik." Leipzig, p. 207. *Also* his "Vorlesungen über photographische Optik." Leipzig, 1905, p. 51.

8. Dispersion by a Prism of Small Angle.

Referring to Fig. 4, consider PA to be a ray of white light. At A it is resolved into a series of coloured rays. The path ABP' corresponds, we will suppose, to the light of the D line; then the ray corresponding to the C line will fall above, and that corresponding to the F line below this mean line. From equation (5) the former is deviated from the original direction by an amount $(n_C - 1)\sigma$; and the latter by an amount $(n_F - 1)\sigma$. Thus the rays corresponding to the C and F lines will emerge from the prism making with each other an angle $d\omega$, where

$$d\omega = \{(n_F - 1) - (n_C - 1)\}\sigma = (n_F - n_C)\sigma = dn\,\sigma \quad \ldots \quad (6)$$

Ex. 2.

What is the angle between the rays corresponding to the C and F lines after refraction through a prism of the flint glass O. 103, with a refracting angle of 5°?

For this particular kind of glass, we have from the list of Schott and Co., $n_F - n_C = 0{\cdot}017$
hence

$$d\omega = 0{\cdot}017 \times 5° = 0{\cdot}085 \times 60' = 5'{\cdot}1.$$

9. Achromatic Prisms.

If two prisms of small angle are arranged as in Fig. 5 with their refracting angles opposed, the resultant dispersion produced is equal to the difference between the dispersions produced by the separate prisms. If the angles are σ_1 and σ_2 respectively, and the various characteristics of the two prisms are designated by suffixes 1 and 2, then from equation (6), the resultant dispersion produced is

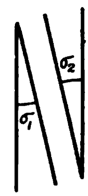

FIG. 5.

$$d\omega = \sigma_1\,dn_1 - \sigma_2\,dn_2 \quad \ldots \quad \ldots \quad (7)$$

If $d\omega = 0$, then there is no angle between the emerging rays corresponding to the F and C lines; that is, they emerge parallel to each other. The rays corresponding to the other Fraunhofer lines also emerge very nearly parallel, so that there is no noticeable colour dispersion. Under these circumstances, the prism is said to be **achromatic**, and

$$\sigma_1\,dn_1 - \sigma_2\,dn_2 = 0 \quad \ldots \quad \ldots \quad (8)$$

is the condition for achromatism.

The *mean* ray, corresponding to the D lines, is deviated (equation (5)) by the small amount

$$\omega = (n_1 - 1)\,\sigma_1 - (n_2 - 1)\,\sigma_2 \quad \ldots \quad (9)$$

where n and n_2 refer to the D lines.

We can rewrite this :

$$\omega = \nu_1 \, dn_1 \, \sigma_1 - \nu_2 \, dn_2 \, \sigma_2 \qquad \ldots \quad (10)$$

from which we obtain by combination with (8)

$$\left. \begin{aligned} \sigma_1 &= \frac{\omega}{dn_1 \, (\nu_1 - \nu_2)} \\ \sigma_2 &= \frac{\omega}{dn_2 \, (\nu_1 - \nu_2)} \end{aligned} \right\} \qquad \ldots \quad (11)$$

giving the refracting angles of two prisms that are to produce the deviation ω, and that are to be achromatic.

Ex. 3.

An achromatic combination is to be made up from crown O. 60 and flint O. 103, so that the mean ray will be deviated by an amount 3^c. What are the angles of the component prisms ?

For crown O. 60, $dn = n_F - n_C = 0{\cdot}0086$; $\nu = 60{\cdot}2$

For flint O. 103, $dn = n_F - n_C = 0{\cdot}0171$; $\nu = 36{\cdot}2$

Hence

$$\sigma_1 = \frac{3^\circ}{0{\cdot}0086 \times 24} = 14^\circ \, 32'$$

$$\sigma_2 = \frac{3^\circ}{0{\cdot}0171 \times 24} = 7^\circ \, 18'{\cdot}5$$

For the benefit of those readers who are not in a position to refer to the German books mentioned by the author, the translators have endeavoured to supplement these by references to books in English dealing with that part of the subject referred to. Thus, for information dealing with the substance of this chapter, *see* Southall "Principles and Methods of Geometrical Optics." Macmillan, 1910. Chap. 13.

Hovestadt "Jena Glass and its Applications." Macmillan, 1902.—Trans.

CHAPTER II.

Image Formation by a Centred System of Spherical Surfaces in the Paraxial Region.

10. Refraction at a Spherical Surface.

We will now investigate the phenomena due to the passage of light from one medium of refractive index n to another of refractive index n', the refracting surface being spherical. The resulting laws are of particular importance, since most optical systems, including that of the eye, are built up of such surfaces, separating differently refracting media. In Fig. 6 let LL_1 be a refracting surface; from its centre, M, draw a straight line MS; this is called an **Optical axis**. On this latter, and in the object space of refractive index n, consider a point P, from which rays are diverging. One of these rays strikes the surface in the point A, and the straight line MAN is the normal at this point. A plane through the points PMA is the plane of incidence (plane of paper), and the refracted ray, according to the law of refraction, must lie in it. This latter ray cuts the optical axis in the point P', and is indicated by the straight line AP'; PAN is the angle of incidence; MAP' the angle of refraction. If the position of P and the angle of inclination APM of the incident ray to the axis be known, then the position of P' may be found quite easily by trigonometry. If this be repeated for a number of rays from P at different inclinations to the axis, we obtain a system of refracted rays, as shown in Fig. 7. Consider a rotation of this system of rays about the optical axis, then these refracted rays now envelope a bright surface with a peak at P'; in this bright surface, the rays overlap and reinforce one another. The property of a plane mirror that all rays from a point P in the object space intersect, after reflection, in a single point, or appear to come

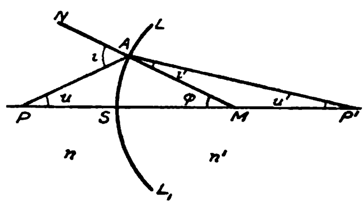

FIG. 6.

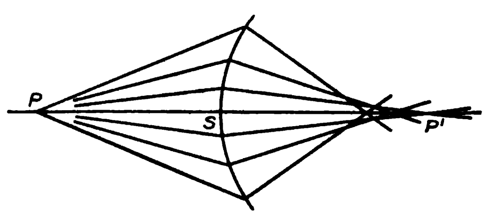

FIG. 7

from such a point, does not hold in this case. Instead of a point image, there appears, on the contrary, a confusion of rays. Thus rays remote from the axis meet the axis, after refraction, in points lying nearer to the vertex S. This phenomenon is called **Spherical Aberration***, and constitutes a defect of optical systems, which optical designers have endeavoured to correct for many years, and with important results. By a suitable choice of the kinds of glass to be used, and by assigning to the surfaces their proper curvatures, this defect may be to a great extent eliminated, so that all the refracted rays pass very approximately through the same axial point. The human eye, consisting essentially of centred refracting surfaces, also suffers from this defect to a small extent. Thus there is no necessity to correct an optical system which is to be used in conjunction with the eye, beyond a certain refinement.†

If we restrict ourselves to rays travelling very close to the axis, the spherical aberration disappears almost entirely. It may then be assumed that all rays from an axial point in the object space intersect in a *single* point in the image space. The space situated in the immediate neighbourhood of the optical axis and thus satisfying the above condition, is called the **Paraxial Region.** With such a restriction, it is desired to find the position of the point P'.

Let

$$N\hat{A}P = i$$
$$M\hat{A}P' = i'$$
$$A\hat{P}S = u$$
$$A\hat{M}P = \phi$$
$$A\hat{P'}M = u' \; ;$$

further

$$SP = a$$
$$SP' = a'$$
$$AM = r$$

If we assume the angles u, u' and ϕ very small, the arc $SA = s$ may be considered as a short line perpendicular to the optical axis, then

$$u = \frac{s}{a}, \quad u' = \frac{s}{a'}, \quad \phi = \frac{s}{r} \qquad \ldots \qquad \ldots \qquad (1)$$

* *See* Chap. V.—Trans.

† The logic of this statement is not convincing. In certain instruments it is quite possible to improve the result by providing a certain amount of *over correction* as compensation for the defects of the human eye referred to.—Trans.

The law of refraction gives

$$n \sin i = n' \sin i''$$

Since the angles i and i'' are small, we may replace the sines by the circular measures of the various angles involved, and hence obtain :

$$n\, i = n'\, i'' \qquad \ldots \qquad \ldots \qquad (2)$$

From the triangles PAM, $P'AM$ we obtain :

$$i = u + \phi \text{ and } i'' = \phi - u'$$

and substituting these values for i and i'' in equation (2) :

$$n\ (u + \phi) = n'\ (\phi - u')$$

i.e.

$$nu + n\phi = n'\phi - n'u'$$

or

$$nu + n'u' = n'\phi - n\phi$$

and

$$nu + n'u' = \phi\ (n' - n)$$

Introducing the values for u, u' and ϕ from equation (1) and dividing throughout by s, we obtain

$$\frac{n}{a} + \frac{n'}{a'} = \frac{n' - n}{r} \qquad \ldots \qquad \ldots \qquad (3)$$

This equation will be called the *Fundamental Equation.* If the values n, n' and r (the constants of the system) be known, and further, the distance a of the luminous point P from the vertex S of the refracting surface, then the magnitude of a', the distance of the image from the vertex may be easily calculated. Since equation (3) contains no terms of angular magnitude, a' is independent of the sizes of the angles u, u', i, i'' and ϕ. This simply means that the point P' is fixed no matter what the value of u be, provided it be small. The last equation also states that all rays from P, inclined at a small angle with the axis, intersect in a single point, namely the image, P'. We have thus obtained a mathematical expression for the paraxial region. An object point and its corresponding image point are called **conjugate foci** or **conjugate points.**

If the point P move along the axis, whilst the values of n, n' and r remain unchanged, then a' varies according to equation (3). Consider two particular positions of the conjugate foci P and P'.

(1) When P lies to the left at infinity ; then $a = \infty$ and $\frac{n}{a} = 0$. In this case, P$'$ is at a distance SP' from

the vertex S, this distance, represented by f', being called the **Second Focal Distance.** P' lies at the **Second Focal Point,** and is generally designated by F'. Under these circumstances equation (3) becomes

$$\frac{n'}{f'}=\frac{n'-n}{r}$$

from which

$$f'=\frac{n'r}{n'-n} \qquad \ldots \qquad \ldots \qquad (4)$$

(2) If now the point P approach the refracting surface, the point P' will move to infinity for a certain distance PS. This distance is then called the **First Focal Distance,** and is usually represented by f. The point P in this case is called the **First Focal Point,** and is designated by F. For the magnitude of f we put $a'=\infty$ and $a=f$ in equation (3) and obtain :

$$\frac{n}{f}=\frac{n'-n}{r}$$

or

$$f=\frac{nr}{n'-n} \qquad \ldots \qquad \ldots \qquad (5)$$

Suppose now equation (3) be multiplied by r and divided by $(n'-n)$, then we obtain

$$\frac{nr}{n'-n}\cdot\frac{1}{a}+\frac{n'r}{n'-n}\cdot\frac{1}{a'}=1 \qquad \ldots \qquad (5a)$$

which gives, from equations (4) and (5)

$$\frac{f}{a}+\frac{f'}{a'}=1 \qquad \ldots \qquad \ldots \qquad (6)$$

Further, dividing equation (5) by (4), we obtain

$$\frac{f}{f'}=\frac{n}{n'} \qquad \ldots \qquad \ldots \qquad (7)$$

giving the general rule :

The two focal distances of a refracting spherical surface are proportional to the refractive indices of the media containing them.

Ex. 4.

If the first medium is air and the second glass of refractive index $n'=1\cdot5$, where do the focal points lie ?

In this case $n=1$.

From (4) :

$$f'=\frac{1\cdot5\,r}{1\cdot5-1}=3r$$

and from (5):

$$f = \frac{r}{1 \cdot 5 - 1} = 2r$$

Thus the first focal distance is double, and the second three times the radius. Fig. 8 shews the positions of the foci F and F'.

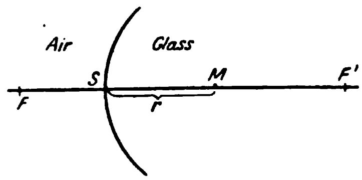

FIG. 8.

Ex. 5.

Where does the image of a point 10 cm. to the left of S lie; the radius of the sphere between air and glass being 8 cm.?

In equation (3), $n = 1$, $n' = 1 \cdot 5$, $r = 8$ cm., and $a = 10$.

Hence

$$\frac{1}{10} + \frac{1 \cdot 5}{a'} = \frac{1 \cdot 5 - 1}{8}$$

or

$$\frac{1 \cdot 5}{a'} = \frac{0 \cdot 5}{8} - \frac{1}{10} = -\frac{3}{80}$$

$$\therefore a' = -40 \text{ cm.}$$

From the negative sign we conclude that the image is virtual and lies 40 cm. to the left of S.*

Ex. 6.

If the first medium is air and the second (on the right) has refractive index n, where will rays incident parallel to the axis meet, if the radius of the spherical surface be r?

From equation (4), $n = 1$, $n' = n$.

Then $f' = \frac{nr}{n-1}$. Thus at the distance f' to the right of the vertex lies the focal point.

Ex. 7.

Let the first medium have refractive index n and the second be air. A converging bundle of rays is incident on the spherical surface of radius r. Where will the image lie if the rays are converging on a point distant r to the right of the pole?

In equation (3) we put

$$n = n, \; n' = 1, \; a = -r$$

---

* The reason for such an assumption will be seen later in the discussion of the sign convention.—Trans.

and obtain

$$-\frac{n}{r}+\frac{1}{a'}=\frac{1-n}{r} \text{ or } a'=r$$

Hence object and image coincide at the centre M of the surface.

Ex. 8.

Where does the image point lie if the object is at the vertex S of the refracting surface ?

Here $a=0$. Putting equation (6) in the form

$$f\,a'+a\,f'=aa'$$

we obtain

$$a'f=0 \text{ or } a'=0$$

Therefore object point and image point coincide at the vertex.

Ex. 9.

The human eye, from which the crystalline lens is removed (termed the aphakic eye), can be assumed to consist of a simple refracting surface (the cornea). What are the focal distances of such an eye if the radius of the cornea be 8 mm. and the refractive index of the medium of the eye is taken to be $\frac{4}{3}$?

In equation (4) put $n=1$, $n'=\frac{4}{3}$, $r=8$ mm. and

$$f'=\frac{4\times 8}{3\,(\frac{4}{3}-1)}=32 \text{ mm.}$$

The first focal distance is obtained most easily from equation (7) ; thus

$$f=\tfrac{3}{4}\,f'=24 \text{ mm.}$$

Ex. 10.

If the length of the aphakic eye is 22 mm., at what distance will it see an object clearly ?

In equation (3) put $n=1$, $n'=\frac{4}{3}$, $r=8$ mm., $a'=22$ mm. ; then

$$\frac{1}{a}+\frac{4}{3}\times\frac{1}{22}=\frac{\frac{4}{3}-1}{8}$$

or

$$\frac{1}{a}=\frac{1}{24}-\frac{2}{33}\,; \qquad a=-\,52{\cdot}8 \text{ mm.}$$

The negative sign of a means that the object lies to the right. Thus the aphakic eye, as a rule, does not see external objects distinctly. It can focus to a sharp image only those rays which are convergent.

11. Image Formation of an object perpendicular to the axis.

In Fig. 9, let PQ be an object of small dimensions perpendicular to the axis. In the same way that all paraxial rays from P are united in the point P', so are all rays from Q united after refraction in Q', where Q' is the image of Q, and lies on the straight line QM produced, M being the centre of the surface. This line is the optical axis with

reference to Q, just as PM is the axis referred to P. Then the point Q' must lie on this axis; from symmetry, since PQ is so small, MP and MQ are equal and so are MP' and MQ'.

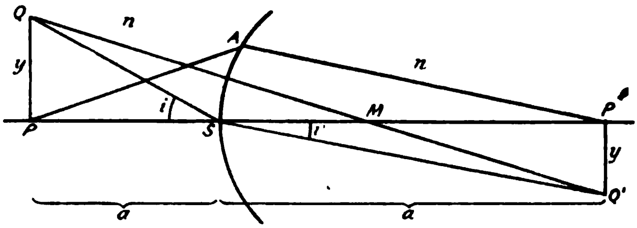

FIG. 9.

Thus the image $P'Q'=y'$, of the object $PQ=y$, is also perpendicular to the axis. The planes perpendicular to the axis in which lie the object and its corresponding image are known as **conjugate planes.** If, from all the rays leaving Q, we choose the one that passes through S, then this ray also, after refraction, must pass through Q'. $Q\hat{S}P=i$ is the angle of incidence, and $Q'\hat{S}P'=i'$ is the angle of refraction, and from the law of refraction

$$ni=n'i'$$

From the triangles QPS and $Q'P'S$

$$\frac{y}{a}=i \text{ and } \frac{y'}{a'}=i'$$

Thus

$$n\frac{y}{a}=n'\frac{y'}{a'}$$

or

$$\beta=\frac{y'}{y}=\frac{na'}{n'a} \quad \dots \quad \dots \quad \dots \quad (8)$$

from which equation, and with the help of equation (3) § 10, y' can be calculated when y, a and the constants of the system are given. The quotient $\frac{y'}{y}=\beta$ gives the **Lateral Magnification** in conjugate planes.

Ex. 11.*

For which conjugate planes is the image equal in size to the object and on the same side of the axis (*i.e.* erect)?

* The two conjugate planes referred to in this exercise are of extreme importance, as will be seen later (§ 15). With regard to putting $y'=-y$—to specify completely any distance, both magnitude and direction are involved. Thus, in general, if a distance be positive in one direction it is negative in the opposite direction. In this book distances *above* the axis are *positive* in the object space, whilst distances *below* the axis are *positive* in the image space. Thus for an *inverted* image, $y'=+y$. If $y'=-y$, then the image is *erect*.—Trans.

In equation (8) we have to put $y'=-y$ so that $na'=-n'a$. Substituting this value of a' in equation (5a) we obtain $a=a'=0$; thus the two conjugate planes coincide at the vertex of the surface. (Principal planes.)

12. Lagrange's Law.

All the rays from P within the paraxial region pass through the image point P'; let PA, Fig. 9 be one of these rays The incident and refracted rays make with the axis the angles u and u'. Then

$$u=\frac{s}{a} \text{ and } u'=\frac{s}{a'}$$

Inserting these values for a and a' in equation (8) we obtain

$$n\,u\,y=n'\,u'\,y' \quad \dots \quad \dots \quad (9)$$

a law first enunciated by the mathematician Lagrange, and which will be found most useful later. In words:

The product of the refractive index, size of object, and inclination of ray to the optical axis, remains unaltered after refraction at a surface.

13. Second Proof of the Fundamental Equation.

In Fig. 9, a ray from Q through M must pass through Q' since the ray QM intersects the surface normally and therefore suffers no refraction. Further

$$PM=a+r \qquad\qquad P'M=a'-r$$

The triangles PQM and $P'Q'M$ are similar, and hence

$$\frac{y'}{y}=\frac{a'-r}{a+r} \quad \dots \quad \dots \quad \dots \quad (9a)$$

Combining this with equation (8), we have

$$\frac{na'}{n'a}=\frac{a'-r}{a+r}$$

which is equation (3) § 10 in another form.

14. Sign and Magnitude of Curvature. Reflection a special case of refraction.

If, in Fig. 6, the centre M of the surface move to the right, the radius increases and the curvature decreases. If M move to infinity, the part of the infinitely large spherical surface in the neighbourhood of the vertex S may be considered as a plane. The two media of refractive indices n and n' are thus separated by a plane through S perpendicular to the axis.

For $r=\infty$, equation (3) becomes

$$\frac{n}{a}+\frac{n'}{a'}=0$$

or

$$\frac{a'}{a}=-\frac{n'}{n} \qquad \dots \qquad \dots \qquad (10)$$

and equation (8) gives

$$\frac{y'}{y}=-1 \text{ or } y'=-y$$

and if y is taken to be positive (erect), Equation (10) shows that the image y' is virtual; the last equation shows that it is of the same size as the object and also erect; there is no magnification.

Ex. 12.

Where does an eye in air see an object which is placed 1 m. under water?

In Fig. 10 let the medium on the left be water and that on the right be air ($n'=1$) Further $a=1$ m. From equation (10)

$$a'=-\tfrac{3}{4}\,m=-75 \text{ cm (for water } n=\tfrac{4}{3})$$

The negative sign indicates that the image lies to the left of S. In Fig. 10 the object PQ appears to have moved nearer to the eye of the observer by a distance PP' = 25 cm.

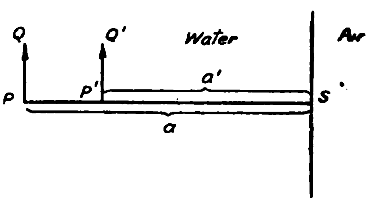

FIG. 10.

If the centre M of the surface lie to the left of S, then the curvature changes sign. The surface now presents its **concave** side to the incident light, whereas in the previous case its **convex** side was turned towards the incident light. To reach M from the point S, the origin of measurement, we now move in the opposite direction, hence the radius takes the opposite (*i.e.* the negative), sign. Hence the rule:

Surfaces which present their convex side to the incident light are to be considered as positive surfaces; those which present their concave side, as negative surfaces.

For the case of reflection of light at a spherical surface, we have merely to put

$$n=-n'$$

in the general relations derived for refraction at a surface; only it must be borne in mind that in cases of reflection, the distances in the image space are to be reckoned positive when

measured in the *opposite* direction to that of the incident light; that is, positive in the same direction as the reflected light.

If, for example, we wish to find the relation between the distances for a concave mirror, we have to replace the magnitudes n, a', and r, in equation (5a) by $-n'$, $-a'$, and $-r$, obtaining

$$\frac{1}{a}+\frac{1}{a'}=\frac{2}{r} \quad \dots \quad \dots \quad (10a)$$

and equation (8) for the lateral magnification gives

$$\frac{y'}{y}=\frac{a'}{a} \dots \quad \dots \quad \dots \quad (11)$$

15. Image Formation by a Centred System of Spherical Surfaces.

By a centred system is understood a system of spherical surfaces whose centres all lie on the same straight line, the optical axis. The points in which the optical axis cuts these spherical surfaces are called the vertices. Between these spherical surfaces we conceive optical media of different refractive indices to exist. The first medium may have a different refractive index from the last medium; thus it is not necessary for the entire system to be surrounded by air. Telescopes, microscopes, magnifying glasses, photographic objectives and the eye are examples of centred systems.

We have seen that in the paraxial region, all rays from a point in the object space unite accurately in a single point after refraction at a spherical surface. It is clear that if the rays which form the image point after the refraction at the first surface are incident on a second surface, the same phenomenon must occur, *i.e.*, these rays after the second refraction will again unite in a single point. This will occur as many times as there are refracting surfaces, and ultimately, after the last refraction, an image point will be formed through which pass all those rays from the object point in the first medium. We can state this rule :

All paraxial rays from an object point in the first medium of a centred system of refracting spherical surfaces unite, after refraction through the system, accurately in a point which is called the image point, or the point conjugate to the original object point.

Further, we have seen that in the case of refraction of light at a single spherical surface, all points lying in a plane perpendicular to the axis are imaged, after refraction,

in a plane also perpendicular to the axis. Such planes we have already called conjugate planes. It was seen also that of any given figure in the object plane an exactly similar figure is formed in the image plane, *i.e.* all angles of the figure in the object plane remain unchanged in the image figure and only the linear dimensions differing. The relative sizes of the straight lines depend upon the positions of the two conjugate planes, and with any given pair of such planes the ratio of image to object is the same over the whole extent of these planes. It is clear that the last result must hold for any centred system, because, after refraction at each succeeding surface, there will be another conjugate plane, the image in which will be similar to the preceding one; only the ratio of the sizes of these images, *i.e.* the lateral magnification changes. Hence

To any plane perpendicular to the axis in the first medium of a centred system there is in the last medium a conjugate plane perpendicular to the axis, the image figure in the latter being similar to the object figure in the former.

In the case of image formation at a single spherical surface, we found that there exist two focal points and focal planes. The general system under consideration will also have two focal points; for if we consider in the first medium a bundle of rays parallel to the axis, these rays will unite, after refraction at the first surface, in a point. This point will be 'imaged' by the second surface and so on, until finally there is formed in the last medium an image point which is the focal point of the whole system, since all rays parallel to the axis in the first medium unite in this point. Suppose now the system be reversed and we consider a beam of rays from the last medium parallel to the axis; then in exactly the same way we may demonstrate the existence of the focal point lying in the first medium, *i.e.*, the first focal point. This embodies the principle of the reversibility of light. A bundle of rays from the first focal point, after refraction emerges parallel to the axis in the last medium. The first and second focal points are represented as before by F and F' respectively. Planes through these points perpendicular to the axis are called respectively the first and second **focal planes.** The second focal plane (*i.e.*, the F' plane) is conjugate to an infinitely distant plane in the object space, whilst the first focal plane (the F plane) is conjugate to an infinitely distant plane in the last medium—the image space.

In Example 11 we had occasion in dealing with refraction at a spherical surface to mention two particular conjugate

planes called the principal planes. In these planes the object and its corresponding image were equal in size and on the same side of the axis.

By the **principal planes** of a system is understood such conjugate planes in which the object and image are equal in size and on the same side of the axis. From Example 11 it was found that for a single refracting surface the principal planes coincide at the vertex of the surface. The magnification in the principal planes is numerically equal to unity. This fundamental property of principal planes must be strongly emphasized, viz.: that **all rays from a point in the first principal plane unite after refraction, in a point in the second principal plane which is the same distance above or below the axis as the point in the first principal plane.**

In Fig. 11 the principal points of the system on the optical axis are denoted by H and H', and the respective focal points by F and F'. Through these four points pass four planes, the two principal planes and the focal planes, by means of which one is able to determine the position of the image of an object PQ lying in the first medium. Thus, all rays from the point Q in the first medium must, after refraction through the system, intersect in the required image point Q'. Of the infinite number of rays from Q it is only necessary to choose two which may be easily traced to the last medium by employing the above properties of the focal points and the principal planes. These two rays are:

1. One ray leaving Q parallel to the axis.
2. The other ray leaving Q and passing through the first focal point F.

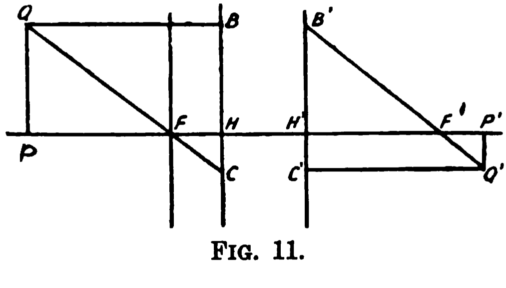

FIG. 11.

Let the first ray cut the H-plane (the first principal plane) in the point B. Since this ray passes through the point B, no matter how many refractions it may have suffered, it must enter the last medium through the conjugate

point B' in the H'-plane (the second principal plane), this point B' being the same distance from the axis as B; *i.e.*, $BH = B'H'$ (Fig. 11). In this way the point B' is found, through which the ray from Q enters the last medium. Since the ray QB in the first medium was parallel to the axis, it must pass through the principal focus F' in the last medium. Hence the two points B' and F' completely determine the ray in the last medium.

Consider now the second ray from Q passing through F, and let it cut the H-plane in C. As before, the conjugate ray in the image space must pass through C' (where $HC = H'C'$), no matter how many refractions the ray may suffer. Since the ray passed through the first principal focus F in the first medium, it must emerge parallel to the axis in the last. Hence this second ray is fully determined. If the ray $B'F'$ be now produced until it cuts the ray from C' parallel to the axis, in the point Q', then Q' is the image of Q. In this way it is possible to construct easily for any possible position of the object PQ the corresponding image $P'Q'$.

A system such as is shown in Fig. 11 acts as a convex lens. For a distant object PQ, an inverted diminished image will be formed. If the object approach the first focal plane the image will rapidly increase in size. When PQ reaches the first focal plane the image will lie at infinity. When the object lies immediately within the first focal distance, the image becomes virtual and very large but diminishes with a further advance of the object. If the latter reach BH in the first principal plane, the image lies at $B'H'$ in the second principal plane, and is of the same size and lies on the same side of the axis as the object.

Fig. 11 not only supplies us with a construction for the image of a given object perpendicular to the axis, but from it, we derive in a simple manner the algebraic relations between the positions of the object and its image. As before, the focal distances FH and $F'H'$ will be designated by f and f'. The reason for so doing is not at once obvious, but will be seen by what follows.

We can now make the following statement for a general system:

The first focal distance is the distance of the first focal point from the first principal point.

The second focal distance is the distance of the second focal point from the second principal point.

16. Conjugate Distances and the Magnification in Conjugate Planes, referred to the Focal Points.

In Fig. 11 the triangles PQF and FHC are similar. Putting

$$QP=y \qquad HC=y' \text{ and } PF=x$$

we have

$$\frac{y'}{y}=\frac{f}{x} \qquad \ldots \qquad \ldots \qquad (11a)$$

Similarly from the similar triangles $F'P'Q'$ and $B'F'H'$, if we put $P'F'=x'$ and since $H'F'=f'$, $B'H'=y$ and $P'Q'=H'C'=HC=y'$, we have

$$\frac{y'}{y}=\frac{x'}{f'} \qquad \ldots \qquad \ldots \qquad (12)$$

The distances x and x' are the axial distances of the conjugate points P and P' measured from the corresponding focal points. The dimensions y and y' are the sizes of the object and image in the conjugate planes laid perpendicular to the axis through P and P'. The ratio $\frac{y'}{y}$ has already been represented by β in the case of a single refracting surface, and is termed the **Lateral Magnification.** We thus have

$$\textit{magnification } \beta=\frac{f}{x}=\frac{x'}{f'} \qquad \ldots \qquad (13)$$

from which is obtained

$$xx'=ff' \ldots \qquad \ldots \qquad \ldots \qquad (14)$$

from which equation the distance x' may be calculated from the object distance x, if f and f' be known.

For example, in the human eye let $f = 15$ mm., $f'=20$ mm., then $ff'=300$ and therefore $xx'=300$.

Ex. 13.

An object 1 m. high (a child) is situated 15 m. from the first focal point F of the eye of an observer; what is the size of the image formed by the eye, and how far does the image lie from the second focal point F' of the eye?

Care must be exercised in all dioptric calculations to express all lengths in the same units, in this case, say mm. Hence

$$x=15000 \text{ mm.} \qquad y=1000 \text{ mm.}$$

Then from equation (11*a*) $\frac{y'}{1000}=\frac{15}{15000}$, *i.e.*, $y'=1$ mm.

From equation (14) $15000\,x'=15\times 20$ or $x'=\frac{1}{50}$ mm.

Hence the image is formed about $\frac{1}{50}$ mm. from the second focal point F'.

17. Conjugate Distances referred to the Principal Points.

If in Fig. 11 we put

$$PH = a \text{ and } P'H' = a'$$

then

$$x = a - f \text{ and } x' = a' - f' \quad \dots \quad (14a)$$

Equation (14) § 16 then becomes

$$(a-f)(a'-f') = ff'$$

or

$$aa' - af' - a'f + ff' = ff'$$

or

$$aa' = af' + a'f$$

Dividing throughout by aa' we obtain

$$\frac{f}{a} + \frac{f'}{a'} = 1 \quad \dots \quad \dots \quad (15)$$

18. Lagrange's Law for a Centred System.

It has already been observed that the product of the size of object, refractive index, and the angle of inclination with the axis of a ray from the axial point of the object, does not alter after refraction at a single surface. If y is the size of the object, n the refractive index and u the corresponding inclination with the axis, then

$$n\,u\,y = n'\,u'\,y'$$

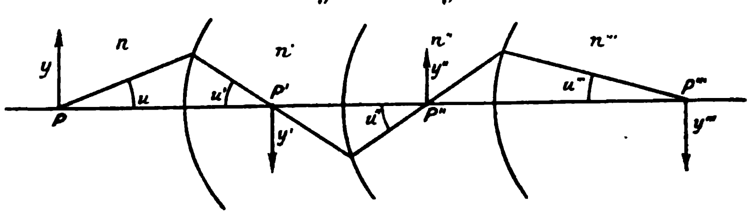

FIG. 12.

where the indexed letters have the same significance *after* refraction as single letters have before refraction. Let Fig. 12 represent a system of centred surfaces and let a ray start from P in the first medium with an inclination u. After the first refraction it cuts the axis in P', making an angle u' with the axis. The same ray is refracted again at the second surface, cuts the axis at P'' and falls on the third surface and so on. Let $y, y', y''\dots$, represent the sizes of the successive objects and images situated at the points $P, P', P''\dots$, whilst $n, n', n''\dots$, are the refractive indices of the successive media. Then for refraction at the first surface.

$$n\,u\,y = n'\,u'\,y$$

For refraction at the second surface,

$$n' \, u' \, y' = n'' \, u'' \, y''$$

and at the third surface

$$n'' \, u'' \, y'' = n''' \, u''' \, y'''$$

From these equations it is easily seen that

$$n \, u \, y = n''' \, u''' \, y'''$$

Thus it follows from the last equation that after any number of refractions, the product under consideration remains numerically unchanged, and that equation (9) § 12 holds quite generally for any centred system in which the unaccented letters refer to the object space and the accented letters to the image space.

19. Angular Ratio.*

If a ray start from the axial point of an object y perpendicular to the axis, making an angle u with the axis and suffer any number of refractions through the surfaces of the system, passing finally in the image space through the axial point of the image y', at an inclination u' to the axis, then by Lagrange's Law

$$n \, u \, y = n' \, u' \, y' \quad \ldots \quad \ldots \quad (16)$$

where n and n' are the refractive indices in the object and image spaces, and y and y' the sizes of the object and image respectively.

The latter equation may be written:

$$\frac{n'}{n} \cdot \frac{y'}{y} \cdot \frac{u'}{u} = 1$$

The quantity $\frac{y'}{y}$ is equal to β. Further let

$$\frac{u'}{u} = \gamma.$$

Then γ is called the "**Angular Ratio**" at the two conjugate points (the axial points of y and y'). The above equation becomes

$$\frac{n'}{n} \cdot \beta \cdot \gamma = 1$$

or

$$\frac{u'}{u} = \gamma = \frac{1}{\beta} \cdot \frac{n}{n'} \quad \ldots \quad \ldots \quad (16a)$$

---

* Also called Convergence Ratio—an important quantity in the theory of optical instruments.—Trans.

For the principal planes, object and image are equal in size and on the same side of the axis. From the Normal Figure 12*, it will be seen that the image is inverted after one refraction, erect after two, inverted again after three and so on. If we do not know how many surfaces the system contains, we cannot decide whether the quantity β has the value -1 or $+1$. If the Normal Figure 11 be regarded as the general figure for all cases, then for the principal planes $\beta=-1$ *i.e.*, $B' H'=B H$, because in this figure we have taken the inverted image y' as positive.

Hence for the principal planes

$$\gamma=\frac{u'}{u}=-\frac{n}{n'} \quad \ldots \quad \ldots \quad (17)$$

Fig. 13 shows the two principal planes through H and H'. Let $PQ = y$ be any object whose image is $P'Q'$. A ray from Q through H must pass after refraction, through H' and Q', since H and H', and also Q and Q', are pairs of conjugate points. The angular ratio at the two principal points is now given by the quotient of the two angles QHP and $P'H'Q'$. But it must be observed, however, that in comparison with the Normal Figure 12 these angles u and u' are measured in opposite directions.

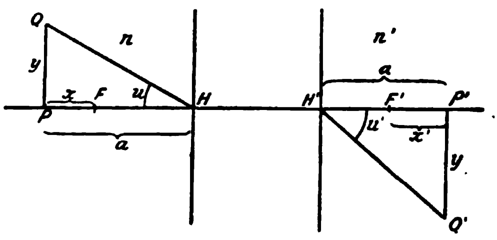

FIG. 13.

Equation (17) states that the angles shown in Fig. 13 are inversely proportional to the refractive indices. Ignoring the negative sign implied in Fig. 13 we have in this case,

$$\frac{u'}{u}=\frac{n}{n'}$$

or

$$nu=n'u'$$

The last expression is simply the Law of Refraction for small angles.

We may express this law thus:

A ray which is directed towards the first principal point in the object space passes into the image space through the

* *See* Appendix to Chapter 5.—Trans.

second principal point as if it had suffered only one refraction at the bounding surface between the two media with refractive indices of the object and image spaces respectively.

20. Magnification in Conjugate Planes, referred to the Principal Points.

From Fig. 13 it follows that

$$u=\frac{y}{a} \qquad u'=\frac{y'}{a'}.$$

Substituting these values in the last equation we obtain

$$\frac{ny}{a}=\frac{n'y'}{a'}$$

and therefore

$$\beta=\frac{y'}{y}=\frac{na'}{n'a} \qquad \dots \qquad \dots \qquad (18)$$

This equation expresses the magnification β in terms of the distances of the conjugate points from the principal points.

21. Relation between the Focal Distances.

From equation (14*a*) §17 we have $a=x+f$ and $a'=x'+f'$. We have already found that $\beta=\frac{f}{x}$, hence the last equation becomes

$$\frac{n\,(x'+f')}{n'(x\;+f)}=\frac{f}{x}.$$

Multiplying both sides of the equation by x, we obtain

$$\frac{n(xx'+xf')}{n'(x+f)}=f.$$

Since

$$xx'=ff'$$

we may write

$$\frac{n(ff'+xf')}{n'(x+f)}=\frac{nf'(f+x)}{n'(x+f)}=f$$

this yielding

$$\frac{f}{f'}=\frac{n}{n'} \qquad \dots \qquad \dots \qquad \dots \qquad (19)$$

Consequently we have the following important rule :—

The first focal distance of a system is to the second focal distance as the refractive index of the object space is to that of the image space.

22. Image of an Infinitely Distant Object. A New Definition for the Focal Distance.

In Fig. 14 an infinitely distant object subtends at the first principal point the visual angle of inclination u.* Then a ray drawn from the outermost point of the object towards H makes with the axis the angle u. This ray after passing

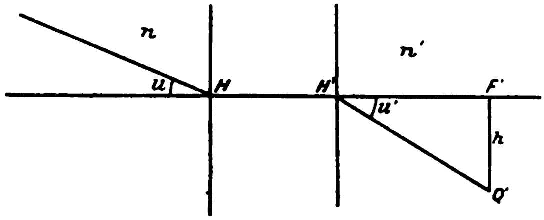

FIG. 14.

through the system emerges through the second principal point H' inclined to the axis at an angle u' so that

$$n u = n' u'$$

Since $H'F' = f'$, then $u' = \frac{h'}{f'}$ if we represent the image of the distant object (in the focal plane) by h'. From the last equation

$$n' u' = \frac{n' h'}{f'}$$

or

$$n u = \frac{n' h'}{f'}$$

and since

$$\frac{n}{n'} = \frac{f}{f'}$$

we have

$$h' = f u \qquad \dots \qquad \dots \qquad (20)$$

The size of the image in the second focal plane of a very distant object is equal to the first focal distance of the system multiplied by the visual angle in the object space. The first focal distance is equal to the size of the image of an infinitely distant object divided by the apparent angular magnitude of the distant object.

Ex. 14.

An optical system forms an image 7 mm. high of a distant object which subtends an angle of 6° at the first principal point. What is the first focal distance of the system ?

---

* Or apparent angular magnitude.—Trans.

From equation (20), $f = \dfrac{h'}{u}$ Since $h' = 7$ mm. and u in circular measure is 0·1047, we have

$$f = \frac{0\ 7}{0{\cdot}1047} = 6{\cdot}69 \text{ cm.}$$

23. Nodal Points.

There are two conjugate points, called **Nodal Points,** which have the property that a ray which in the object space is directed towards the first nodal point, passes in the image space through the second nodal point parallel to its original direction. The angular ratio for these two points must therefore be equal to unity. In order to locate the positions of these points, since $\gamma = -1$, equation (16a) § 19 becomes :

$$\beta = -\frac{n}{n'} = -\frac{f}{f'}$$

and from equations (11a) and (12) §16 we obtain

$$\beta = \frac{f}{x} = -\frac{f}{f'} = \frac{x'}{f'}$$

Hence

$$x = -f' \text{ and } x' = -f$$

The first nodal point is distant f' from F and the second nodal point is distant f from F'.

24. Combination of Two Systems.

The first focal distance of a system is expressed by the quotient of the size of the image in the second focal plane of an infinitely distant object divided by the apparent angular magnitude of this distant object. [Equation (20) § 22.]

This definition also holds for a combination of two systems. To determine the first focal distance of such a system it is only necessary to find the size of the image of a very distant object and divide by the visual angle.

In Fig. 15 let H_1 and H_1' be the principal points ; F_1 and F_1' the focal points, f_1 and f_1' the focal distances—all

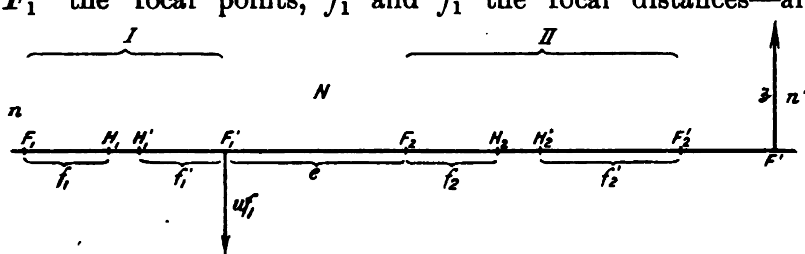

FIG. 15.

referring to the first system. In like manner H_2 and H_2', F_2 and F_2', f_2 and f_2', refer to the second system. We shall designate the two systems by I. and II. respectively.

A very distant object lying on the left hand side will form an image of size f_1u in the F_1'-plane after traversing system I, the visual angle being u. This image in turn becomes the object for system II. For the moment let us call the image formed by system II, z. Since this image lies on the opposite side of the axis as compared with the normal figure it takes the negative sign and consequently equation (11a) §16 becomes

$$-\frac{z}{f_1u}=\frac{f_2}{\overline{F_1'F_2}}$$

where $\overline{F_1'F_2}=e$, the distance between the two adjacent focal points of the two systems. Since $\frac{z}{u}$ is the resultant focal distance f of the combination, we have

$$\frac{z}{u}=f=-\frac{f_1f_2}{e} \quad \ldots \quad \ldots \quad (22)$$

Expressed in words this is :

The first focal distance of a combination of two systems is equal to the product of the first focal distances of the component systems divided by the distance between the two adjacent focal points.

If $H_1'H_2=d$ is the distance between the adjacent principal points of the two systems, and $K_1'K_2=k$ is the distance between the two adjacent nodal points of the respective systems, (K_1' and K_2 are these nodal points and are not shown in the figure) then from Fig. 15 :

$$f_1'+e+f_2=d$$

similarly

$$f_1+e+f_2'=k,$$

and we have for f the two following expressions :—

$$f=-\frac{f_1f_2}{d-f_1'-f_2} \quad \ldots \quad \ldots \quad (23)$$

and

$$f=-\frac{f_1f_2}{k-f_2'-f_1} \quad \ldots \quad \ldots \quad (24)$$

If in Fig. (15) the light is incident from the right we obtain in precisely the same way, the second focal distance of the combination.

$$-f'=\frac{f_1'f_2'}{e}=\frac{f_1'f_2'}{d-f_1'-f_2}=\frac{f_1'f_2'}{k-f_2'-f_1} \quad (25)$$

If F' is the second principal focus of the combination, it follows from equation (14) § 16

$$e \, . \, \overline{F_2'F'} = f_2 f_2'$$

hence

$$\overline{F_2'F'} = \frac{f_2 f_2'}{e}$$

and therefore

$$\overline{H_2'F'} = \frac{f_2 f_2'}{e} + f_2' \, .$$

Since F' is the second principal focus of the combination, we may obtain h', the distance between the second principal point of the combination and the second principal point H_2' of system II., if we subtract the second focal distance f' of the combination from the distance $\overline{H_2'F'}$. Hence

$$h' = \frac{f_2 f_2'}{e} + f_2' + \frac{f_1' f_2'}{e} = \frac{f_2'}{e} (f_2 + e + f_1').$$

But

$$f_2 + e + f_1' = d$$

hence

$$h' = \frac{f_2' d}{d - f_2 - f_1'} \quad \dots \quad \dots \quad (26)$$

Similarly we find h, the distance of the first principal point of the combination from the first principal point H_1 of system I., *i.e.*,

$$h = \frac{f_1 d}{d - f_2 - f_1'} \quad \dots \quad \dots \quad (27)$$

We may express the magnification β of a combination of two systems L_1 and L_2 in the following way :

In Fig. 16, L_1 and L_2 are the two systems with the first and second focal points F_1 and F_1', F_2 and F_2' respectively. The object y_1 at P forms, after refraction through L_1, the image y_1' at P' ; whilst y_1' becomes the object y_2 for the second system L_2. The object y_2 is imaged as y_2' at P'' after refraction by L_2. We have also

$$PF_1 = x_1 \qquad F_1'P' = x_1' \qquad P'F_2 = x_2 \qquad F_2'P'' = x_2'.$$

From equations (11a) and (12) § 16 :

$$\frac{y_1'}{y_1} = \frac{x_1'}{f_1'} \, ; \; \frac{y_2'}{y_2} = \frac{f_2}{x_2} \, ; \; y_1' = y_2$$

also

$$x_1' + x_2 = e$$

The magnification β of the combination follows from these equations

$$\beta=\frac{y_2'}{y_1}=\frac{f_2}{f_1'}\cdot\frac{x_1'}{x_2} \qquad \dots \qquad \dots \qquad (27a)$$

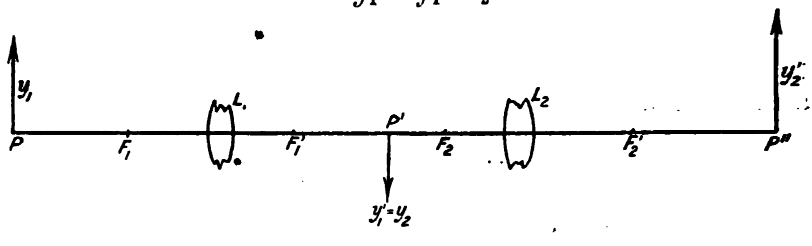

FIG. 16.

If $e = o$, *the two adjacent focal planes of the component systems coincide and*

$$x_1' + x_2 = 0 \quad \text{or} \quad \frac{x_1'}{x_2} = -1.$$

Hence

$$\beta = -\frac{f_2}{f_1'} \qquad \dots \qquad \dots \qquad (27b)$$

In this case the system is said to be **telescopic.** The negative sign of β simply signifies that in the arrangement of Fig. 16 the object at P and the image at P'' are both on the same side of the axis. We can express the rule as follows :

In telescopic systems the magnification is constant and equal to the ratio of the adjacent focal distances of the two component systems.

25. Lenses.

A medium bounded by two (centred) spherical surfaces is called a **lens.** The rules for the formation of images by lenses depend on the results of § 24 if we consider the two separate refracting surfaces as being parts of a combination. According to the positions of these bounding surfaces there are six different forms of lenses which fall into two main groups—**convex** lenses and **concave** lenses—according as rays parallel to the axis are made to converge or diverge after refraction.* Convex lenses are considered positive, concave as negative lenses.

* We here assume, as is generally the case, that the thickness of the lens is small compared with the radii of curvature of the bounding surfaces; otherwise the above classification is incomplete.

Convex lenses may be either bi-convex (Fig. 17), plano-convex (Fig. 18), or concavo-convex (Fig. 19). Lenses of the last kind are usually called positive meniscus lenses. Convex lenses are easily recognized as they are thicker at the centre than at the edge.

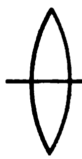

FIG. 17.

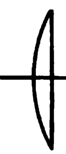

FIG. 18.

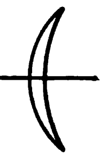

FIG. 19.

Concave lenses may be either bi-concave (Fig. 20), plano-concave (Fig. 21), or of the negative meniscus form (Fig. 22), and are thinner at the centre than at the edge.

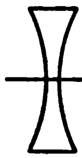

FIG. 20.

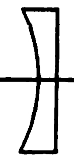

FIG. 21.

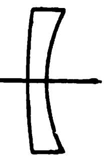

FIG. 22.

If a lens be placed in air, the first focal distance is equal to the second; and for distances referred to the principal points we have, from equation (15), § 17,

$$\frac{1}{a}+\frac{1}{a'}=\frac{1}{f} \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

26. Infinitely Thin Lenses.

For many practical purposes the thickness of lenses may be neglected, *i.e.*, we may assume the two surfaces to coincide; also the two principal points of a spherical surface coincide at the vertex. Hence the value of d in equations (23), (26) and (27) may be put equal to zero. From the last two equations it follows that $h=h'=0$, *i.e.*, **the principal points of a thin lens coincide at the common vertex.**

When $d=0$, equation (23) § 24 becomes

$$f=\frac{f_1 f_2}{f_1'+f_2} \quad \ldots \quad \ldots \quad (29)$$

If the lens be situated in air, whilst the refractive index of the medium of the lens be n, then equation (5) § 10 becomes, if r_1 is the first radius and r_2 the second,

$$f_1=\frac{r_1}{n-1} \qquad f_2=\frac{nr_2}{1-n}$$

and taking into consideration that $f_1'=nf_1$, equation (29) becomes

$$\frac{1}{f}=(n-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)=\frac{1}{a}+\frac{1}{a'} \quad \ldots \quad (30)$$

where the distances a and a' are measured from the common vertex.

For the magnification we have

$$\beta = \frac{y'}{y} = \frac{a'}{a} \qquad \ldots \qquad \ldots \qquad (31)$$

Ex. 15.

What is the focal distance of an equi-bi-convex lens, the radii of its surfaces being r?

In equation (30) if we substitute $r_1 = r$ and $r_2 = -r$, then

$$f = \frac{r}{2(n-1)}$$

If $n = 1 \cdot 5$, then $f = r$.

Ex. 16.

What is the focal length of a plano-convex lens, if the radius of the curved surface is r?

Here $r_1 = r$ and $r_2 = \infty$ and therefore

$$f = \frac{r}{n-1}.$$

For $n = 1 \cdot 5$ we obtain $f = 2r$.

Ex. 17.

What are the focal lengths of a bi-concave and a plano-concave lens respectively with radii r?

In the first case, in equation (30) substituting $r_1 = -r$, $r_2 = r$ we obtain

$$f = -\frac{r}{2(n-1)}$$

In the second, $r_1 = -r$ and $r_2 = \infty$, so that

$$f = -\frac{r}{n-1}$$

27. Combination of Infinitely Thin Lenses.

In Fig. 23 let L_1 and L_2 be two thin lenses with vertices S_1 and S_2. A luminous point P on the axis forms an

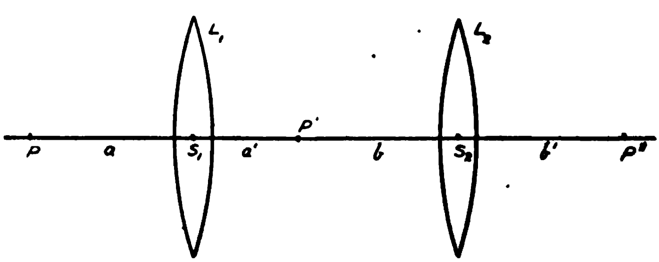

FIG. 23.

image P' after refraction in L_1. P$'$ is the object point for the lens L_2 which forms in the same way the image P''.

Putting $PS_1=a$, $S_1P'=a'$, $P'S_2=b$, $S_2P''=b'$, $S_1S_2=d$, and the focal lengths of the two lenses equal to f_1 and f_2, then:

$$\frac{1}{a}+\frac{1}{a'}=\frac{1}{f_1} \qquad \frac{1}{b}+\frac{1}{b'}=\frac{1}{f_2} \qquad a'+b=d.$$

If we put $d=0$, thus bringing the lenses into contact, $a'=-b$ and the above formulæ become:

$$\frac{1}{a}+\frac{1}{b'}=\frac{1}{f_1}+\frac{1}{f_2} \quad \ldots \quad \ldots \quad (32)$$

in agreement with equation (30) § 26. If f is the focal length of the combination of the two lenses in contact, we have therefore

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2} \quad \ldots \quad \ldots \quad (33)$$

This formula is quite general and consequently we have the rule:

For a system of infinitely thin lenses in contact, the reciprocal of the combined focal length is equal to the sum of the reciprocals of the component focal lengths.

Spectacle lenses are the most important representatives of this type.

28. Construction for the Image formed by a Thin Lens.

For a very thin lens the two focal points F and F' are equidistant from the vertex S (the common principal point) (Fig. 24). To construct the image of an object PQ,

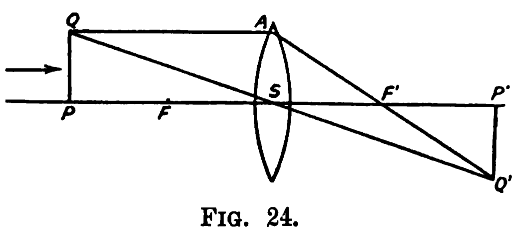

FIG. 24.

draw QA parallel to the axis, cutting the plane of the thin lens in A. From A through the second focal point F' draw a straight line cutting QS produced in the required image point Q'. The line $Q'P'$ perpendicular to the axis is then the image of PQ. In the case of negative lenses, the second focal point F' lies to the left of the lens if the direction of the incident light is from left to right.

CHAPTER III.

Power and Convergence.

29. Power. The Dioptre.

If we use a lens as a simple magnifying glass, it is observed that the glass is "stronger" the shorter the focal length; its strength is inversely proportional to the focal length. A lens having a very long focal length acts more like a simple parallel-sided glass disc, and if its focal length be infinitely great it becomes identical with one. An infinitely great focal length therefore corresponds to a strength or *power* of zero. Hence the reciprocal value of the focal length is called the **power** and is designated by the letter D, *i.e.*,

$$\frac{1}{f}=D \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Expressing the focal length f in metres and putting $f=1$m., the power of such a lens is

$$D=1$$

This unit of power is called a **Dioptre** (Dp.) and a lens having a dioptric strength of unity, has a focal length of 1 metre. Thin lenses, *e.g.*, spectacle lenses having focal lengths of 2, 3 and 4 metres, have dioptric strengths of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ dioptres.

Ex. 18.

What is the power of a lens of 15 cm. focal length?

Here $f=0\cdot15$ m; therefore $D=\frac{1}{0\cdot15}=6\frac{2}{3}$ dioptres (contracted to $6\frac{2}{3}$ *Dp.*).

Ex. 19.

What is the power of a lens of 225 mm. focal length?

We have $f=0\cdot225$ m; therefore $D=\frac{1}{0\cdot225}=4\frac{4}{9}$ *Dp.*

Ex. 20.

What is the power of a lens whose focal length is 10 inches?

$$1 \text{ inch} = 2{\cdot}54 \text{ cm.} = 0{\cdot}0254 \text{ m.}$$

$$\therefore f = 0{\cdot}254 \text{ m. and } D = \frac{1}{0\cdot254} = 3\tfrac{15}{16}\ Dp.$$

Ex. 21.

If the radii of the curved surfaces of a positive meniscus be 15 cm. and 20 cm., what is its power, if the refractive index of the glass be 1·5?

The formula $\frac{1}{f} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$ gives, since $r_1 = 0 \cdot 15$ m and $r_2 = 0 \cdot 2$ m whilst $n = \frac{3}{2}$,

$$D = \tfrac{1}{2}\left(\frac{1}{0 \cdot 15} - \frac{1}{0 \cdot 2}\right) = \tfrac{5}{6}\,Dp.$$

Ex. 22.

A bi-concave lens has radii 7 and 9 inches. If the refractive index of the glass be 1·5, what is its power?

The focal length must be expressed in metres before converting to dioptres.

$$\frac{1}{f} = \tfrac{1}{2}(-\tfrac{1}{7} - \tfrac{1}{9}) = -\tfrac{16}{126} \text{ inch units.}$$

$$f = -\tfrac{126}{16} \text{ ins.} = -\tfrac{126}{16} \times 0{\cdot}0254 \text{ m.}$$

$$\therefore D = -\frac{16}{126 \times 0{\cdot}0254} = -5\ Dp.$$

A very important application of this method of calculation in dioptres is seen in the following rule.

As we have seen above (§ 27), a system composed of a series of very thin lenses (*e.g.*, spectacle lenses) in contact, whose focal lengths are f_1, f_2, f_3, &c., has a resulting focal length of f given by the equation

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \ \&c.$$

Since

$$\frac{1}{f_1} = D_1, \ \frac{1}{f_2} = D_2, \ \&c.,$$

then

$$D = D_1 + D_2 + D_3 + \ \&c. \qquad \ldots \quad (2)$$

The rule is expressed thus :

The power of a series of thin lenses placed in contact is equal to the sum of the powers of the separate lenses.

A lens of power $4\,Dp.$ combined with a lens $5\,Dp.$ gives the same result as a lens of $9\,Dp.$ If a lens of power $+5\,Dp.$ be combined with a lens $-7\,Dp.$ the resultant is a negative lens whose power is $-2\,Dp.$

30. Spectacle Nomenclature.

In earlier days, spectacle lenses, whether bi-convex or bi-concave were classified according to the radii of their curved surfaces expressed in inches and in this way arose the term spectacle numbers. At the present day the *dioptric* strength of a lens is always given.

In Ex. 15 it was found that for a thin bi-convex lens of refractive index 1·5,

$$f = r.$$

In this case the radius of the curved surface gives also the focal length. A spectacle lens whose number was formerly p, would have 0·025 p metres focal length, and consequently its power is

$$D = \frac{1}{0{\cdot}025\, p}$$

hence

$$p.D = 40 \qquad \dots \qquad \dots \qquad (3)$$

The product of the old and new spectacle numbers is therefore equal to 40. If $n = 1{\cdot}528$, which is the value for ordinary plate glass, $f = \frac{r}{1{\cdot}056}$.

31. Convergence

If, in Fig. 25, P be a luminous point on the optical axis, then the light from P incident on the lens will have a certain divergence. Consider the rays PA and PB emanating from P and striking the upper and lower edges of the lens L. It is easily seen that the divergence of these rays is greater as P approaches the lens, and smaller as it recedes from the lens. For an infinitely distant point the divergence is zero. The divergence of a bundle of rays is inversely proportional to the distance of the point P from the vertex S of the lens. We may also have convergent light striking a lens, as for instance when the light has previously traversed a convex lens placed in a suitable position in the path of the rays The divergence or convergence of a bundle of rays is changed in a characteristic manner by refraction through a lens. When the luminous point lies at a considerable distance from the convex lens L, the divergence in the object space is changed by refraction to convergence in the image space, the beam converging on P'. As the luminous point approaches the lens, the convergence in the image space diminishes and reaches a zero value when P coincides with the first focal point. If now the luminous point approach still closer to the lens, the lens is not able to change the great divergence in the object space to convergence. In this case only a diminution of divergence results.

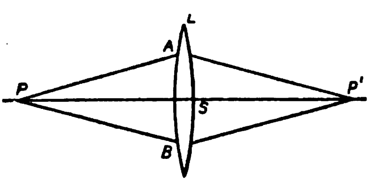

FIG. 25.

Convergence and divergence may be conceived as different portions of a scale of '**Vergence**,' in the same way that high and low temperatures are related to one another on the thermometric scale ; they are merely two forms of vergence, and are separated by a zero point.

Gullstrand has derived the following notation to express the above relationship.

$$\frac{1}{a}=A \qquad\qquad \frac{1}{a'}=A'.$$

where A and A' are the values in the object and image spaces respectively of the vergence, or convergence, since we are dealing with the particular conception of convergence rather than the general idea of vergence.

The formula

$$\frac{1}{a}+\frac{1}{a'}=\frac{1}{f}$$

for a lens in air may now be written in the simpler way.

$$A+A'=D \qquad \ldots \qquad \ldots \qquad (4)$$

We have therefore the rule :

The sum of the vergences before and after refraction is equal to the power of the lens.

Employing this new conception, it is possible to obtain a simple relation for the magnification from the equation

$$\frac{y'}{y}=\frac{a'}{a}.$$

By introducing the vergences A and A' we obtain

$$\beta=\frac{y'}{y}=\frac{A}{A'}, \qquad \ldots \qquad \ldots \qquad (5)$$

i.e., **the magnification is equal to the quotient of the vergences in the object and image spaces.**

The relations become particularly clear when distances in the object space are reckoned in the opposite direction to that which has been employed previously. For example, in the normal figure for the path of rays through a thin lens, if the incident light is considered convergent, then the opposite sign must be given to distances in the object space and consequently to the vergences in the same. Thus instead of equation (4) we obtain

$$A'-A=D \text{ or } A'=A+D.$$

It will be seen from this that A', the vergence after refraction is obtained from A, the vergence before refraction, by adding the power D of the lens as a constant.

Gullsrtand, in his works, adopts the term *Convergence* for the reciprocal values of axial intercepts.*

The importance of equations (4) and (5) lies not only in their simplicity and clearness, but, as Gullstrand points out, in that they are easily applicable to any centred system, and, moreover, for thin pencils which traverse the system at any inclination.†

32. Power and Vergences for any Centred System of Spherical Surfaces.

In conformity with Gullstrand, we *reduce* a distance by dividing it by the refractive index of the medium containing it.

For example, if d be a distance or an intercept along the axis of a system, then

$$\frac{d}{n} = \delta \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

and δ is the **reduced distance** d.

If f be the first focal distance in a medium of refractive index n, then $\frac{f}{n}$ is the reduced focal distance.

Gullstrand defines in general, the **power** of a system, **as the reciprocal of the reduced focal length.**

Hence

$$Power = \frac{n}{f} = D \quad \ldots \quad \ldots \quad (7)$$

but from equation (7) § 10

$$\frac{n}{f} = \frac{n'}{f'} = D \quad \ldots \quad \ldots \quad (8)$$

where f' is the second focal distance and n' the refractive index of the last medium.

---

*In this counry, a corresponding idea exists under a different term, and is used by members of the ophthalmic profession.

If a be an actual distance as measured from the point of reference, then :—

the vergence $\frac{1}{a} = A$, is called *dioptral distance*,

the power $\frac{1}{f} = D$, is called *dioptral number*.

It is easily seen that if a or f be measured in cms. then

$\frac{100}{a} = A$ in dioptres and $\frac{100}{f} = D$ in dioptres.

a or f be measured in inches, then

$\frac{40}{a} = A$ dioptres and $\frac{40}{f} = D$ dioptres.

These form useful rules in practice for the conversion of the common units.—Trans.

† *See*—

Gullstrand " Über die Bedeutung der Dioptrie."

v. Graefes Archiv für Ophthalmologie 1899, p. 46, &c.

Gleichen, Der Mechaniker.

Nicolassee. " Über die Bedeutung der Dioptrie- und Konvergenzrechnung. Berlin, 1909. Nr 14–20.

The power of a system may therefore be derived from the first as well as from the second focal distance.

In the case of refraction at a single surface, equation (5) § 10 becomes

$$D = \frac{n}{f} = \frac{n' - n}{r} \quad \dots \quad \dots \quad (9)$$

In order to obtain the power of a surface we proceed as follows: passing through the surface in the direction of the light, subtract the refractive index of the medium in front of the surface from the index of that behind and divide the difference by the radius expressed in metres.

If a and a' are the axial intercepts of a beam of rays before and after refraction, measured from the principal points, $\frac{a}{n}$ and $\frac{a'}{n'}$ are the reduced intercepts, and therefore their reciprocals $\frac{n}{a}$ and $\frac{n'}{a'}$ are the reduced vergences represented by A and A'. Then

$$A = \frac{n}{a}, \ A' = \frac{n'}{a'} \quad \dots \quad \dots \quad (10)$$

Equation (15) § 17, for any centred system, in which the conjugate distances are referred to the principal points, viz.,

$$\frac{f}{a} + \frac{f'}{a'} = 1$$

since

$$\frac{f}{f'} = \frac{n}{n'},$$

becomes

$$\frac{n}{a} + \frac{n'}{a'} = \frac{n}{f},$$

i.e.,

$$A + A' = D \dots \quad \dots \quad \dots \quad (11)$$

which holds for every centred system.

The sum of the vergences on the object side and image side is thus equal to the power.

For the magnification, equation (18) § 20 gives

$$\beta = \frac{y'}{y} = \frac{A}{A'} \quad \dots \quad \dots \quad (12)$$

whilst for the angular ratio, equation (16a) § 19 yields

$$\gamma = \frac{u'}{u} = \frac{nA'}{n'A}. \quad \dots \quad \dots \quad (13)$$

Consider two component systems:

Let the first system have the first focal distance f_1, and the second distance f_1'. Let the second system have the first focal distance f_2, and the second distance f_2'. The adjacent principal points (the second principal point of the first system and the first principal point of the second) are distant d apart (*see* equation (6) § 32), the reduced value for which is δ, where

$$d = N\delta \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

and N is the refractive index of the medium containing d.

The powers of the two systems are D_1 and D_2. We will assume (as in Fig. 15) that n is the refractive index before the first system, n' that behind the second system, and N that of the medium between the two systems. The space of refractive index N is the image space for system I., and the object space for system II. Consequently we have:

$$\left.\begin{array}{l} D_1 = \dfrac{n}{f_1} = \dfrac{N}{f_1'} \qquad D_2 = \dfrac{N}{f_2} = \dfrac{n'}{f_2'} \\ \text{or} \\ f_1 = \dfrac{n}{D_1},\ f_2 = \dfrac{N}{D_2},\ f_1' = \dfrac{N}{D_1},\ f_2' = \dfrac{n'}{D_2} \end{array}\right\} \quad \ldots \quad (14a)$$

Substituting these values in equation (23) § 24 written in the form:

$$-\frac{f_1}{f} = \frac{d - f_1' - f_2}{f_2}$$

we obtain

$$-\frac{n}{f}\cdot\frac{1}{D_1} = \frac{N\delta - \dfrac{N}{D_1} - \dfrac{N}{D_2}}{\dfrac{N}{D_2}} \text{ or } \frac{n}{f}\cdot\frac{1}{D_1 D_2} = \frac{1}{D_1} + \frac{1}{D_2} - \delta$$

or finally

$$\frac{n}{f} = D_1 + D_2 - \delta D_1 D_2.$$

But

$$D = \frac{n}{f} = \frac{n'}{f'} \quad \ldots \quad \ldots \quad (15)$$

therefore the power of the combination is:

$$D = D_1 + D_2 - \delta D_1 D_2 \quad \ldots \quad \ldots \quad (16)$$

In words:—

The power D of a system consisting of two component systems of powers D_1 and D_2 whose adjacent principal points

are separated by the reduced distance δ, is equal to the sum of the powers D_1 and D_2 minus the product of the powers and the reduced distance δ.

Employing equation (14*a*) § 32 we will now modify equations (26) and (27) § 24.

The denominator of the expressions (26) and (27) § 24 in accordance with equation (23) § 24 is equal to :

$$-\frac{f_1 f_2}{f} = -\frac{n}{D_1}\cdot\frac{N}{D_2}\cdot\frac{D}{n} = -\frac{DN}{D_1 D_2}$$

The numerator of equation (27) § 24 may be written employing equations (14) and (14*a*) § 32

$$f_1 d = \frac{nN\delta}{D_1}$$

Hence

$$h = \frac{nN\delta}{D_1} \div -\frac{DN}{D_1 D_2} = -\frac{n\,D_2\,\delta}{D}$$

Since a distance is reduced by dividing by the refractive index, the distance h in the object space becomes

$$\frac{h}{n} = H$$

hence

$$H = -\frac{D_2\delta}{D} \qquad \dots \qquad \dots \qquad (17)$$

The numerator of expression (26) § 24 becomes

$$f_2' d = \frac{n'}{D_2} N\delta$$

from which

$$h' = \frac{n'}{D_2}.N\delta \div -\frac{DN}{D_1 D_2} = -\frac{n' D_1 \delta}{D}$$

Since the distance h' is in a medium of refractive index n', it is reduced by division, thus :

$$\frac{h'}{n'} = H'$$

and

$$H' = -\frac{D_1\delta}{D} \qquad \dots \qquad \dots \qquad (18)$$

From equations (17) and (18), the positions of the principal points of the combination may be calculated in a particularly simple manner (Gullstrand).

33. Summary of the Results for a System referred to the Principal Points.

The distances a and a', of two conjugate points P and P' (Fig. 26) are measured from the principal points H and H', *positive in the object space if measured* (from right to left) *against the direction of the light, and positive in the image space if measured* (from left to right) *in the same direction as the incident light.**

Then

$$\frac{f}{a}+\frac{f'}{a'}=1 \qquad \ldots \qquad \ldots \qquad \text{(I)}$$

where f and f' are the first and second focal distances respectively.

$$\textit{Magnification} \;=\beta=\frac{y'}{y}=\frac{na'}{n'a} \qquad \ldots \qquad \ldots \qquad \text{(II)}$$

where y' and y are the sizes of the image and object respectively, and n' and n the corresponding refractive indices.

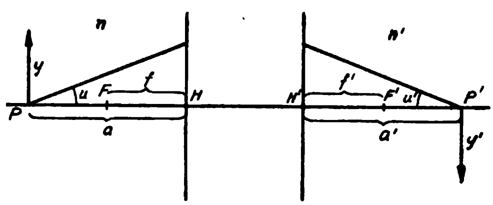

FIG. 26.

$$\textit{Angular Ratio} = \gamma=\frac{u'}{u}=\frac{a}{a'} \qquad \ldots \qquad \text{(III)}$$

where u and u' are the angles made with the optical axis by a ray traversing the system, passing through the axial points of the object and image.

The anterior focal distance is the distance of the anterior focal point from the anterior principal point.

The posterior focal distance is the distance of the posterior focal point from the posterior principal point.

$$\frac{f}{f'}=\frac{n}{n'} \qquad \ldots \qquad \ldots \qquad \text{(IV)}$$

The reduced **vergences** are :

$$A=\frac{n}{a} \qquad\qquad A'=\frac{n'}{a'} \qquad \ldots \qquad \text{(IVa)}$$

* *See* Appendix to Chapter V.—Trans.

The power D of a system is:

$$D=\frac{n}{f}=\frac{n'}{f'} \quad \ldots \quad \ldots \quad \text{(V)}$$

$$A+A'=D \quad \ldots \quad \ldots \quad \text{(VI)}$$

$$\frac{y'}{y}=\beta=\frac{A}{A'} \quad \ldots \quad \ldots \quad \text{(VII)}$$

For a single surface of radius r, which separates a medium of refractive index n from one of refractive index n', we have:

$$f=\frac{nr}{n'-n} \qquad f'=\frac{n'r}{n'-n} \qquad D=\frac{n'-n}{r} \qquad \text{(VIII)}$$

Further, in general

$$z=uf \quad \ldots \quad \ldots \quad \text{(IX)}$$

where z is the image of an infinitely distant object which subtends the visual angle u.

For a combination of two systems with powers D_1 and D_2 we have

$$D=D_1+D_2-\delta D_1D_2 \ldots \quad \ldots \quad \text{(X)}$$

$$H=\frac{h}{n}=-\frac{D_2\delta}{D} \qquad H'=\frac{h'}{n'}=-\frac{D_1\delta}{D} \ldots \quad \text{(XI)}$$

In equations (X) and (XI), D represents the power of the combination, δ the reduced distance between the adjacent principal points of the component systems, H the reduced distance of the first principal point of the combination from that of the first component system ; H' the reduced distance of the second principal point of the second system from that of the whole combination.

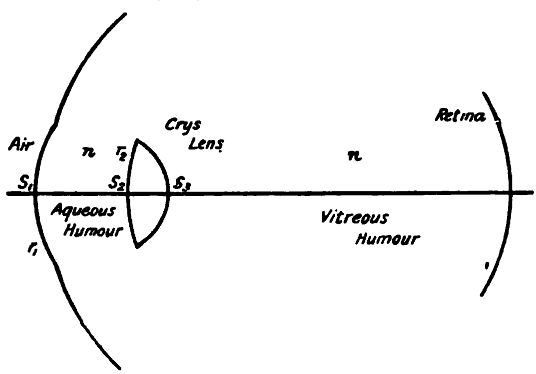

FIG. 27.

In order to illustrate these relations, we will apply them to a number of examples and also to the optical system of the human eye. (*See* Fig. 27.)

The optical system of the eye is not strictly a centred system of spherical surfaces but may be taken as such, to a first approximation. The radii and refractive indices are naturally subject to individual variations; in order to permit of an easy treatment of this subject, various investigators have assumed certain average values, this constituting a "Schematic Eye." For instance, the schematic eye of Helmholtz is based on the following data:

Radius of the cornea at the vertex S_1 ...	$r_1=7{\cdot}829$ mm.
Radius of the first surface of the crystalline lens with vertex S_2	$r_2=10$ mm.
Radius of the second surface of the crystalline lens with vertex S_3	$r_3=-6$ mm.
The thickness of the anterior chamber ...	$S_1S_2=3{\cdot}6$ mm.
The thickness of the lens	$S_2S_3=3{\cdot}6$ mm.
Refractive index of air	$=1$
Refractive index of aqueous humour ...	$n=1{\cdot}3365$
Refractive index of crystalline lens ...	$N=1{\cdot}43$
Refractive index of vitreous humour ...	$n=1{\cdot}3365$

For a more detailed description of the eye see chapter VI § 53.

The later schematic eye of Gullstrand* gives the average values of the actual proportions better than those of Helmholtz. In the following exercises, however, the data of Helmholtz will be employed, as they are more frequently taken as the basis of calculation in ophthalmic literature.

Ex. 23.

What is the power of a surface of radius r separating the air from a medium of refractive index n?

In equation (VIII) putting $n'=n$ and $n=1$ we obtain $D=\dfrac{n-1}{r}$.

Ex. 24.

If we consider the cornea as a single refracting surface of radius $r=7{\cdot}829$ mm., the medium in front being air, that behind being the aqueous humour of refractive index $n=1{\cdot}3365$, what are the values of the focal distances and its power?

In equation (VIII) substituting $n'=n$ and $n=1$, then

$$f=\frac{r}{n-1}=\frac{7{\cdot}829}{0{\cdot}3365}=23{\cdot}26 \text{ mm.}$$

$$f'=\frac{nr}{n-1}=nf=1{\cdot}3365\times 23{\cdot}26=31{\cdot}09 \text{ mm.}$$

---

* *See* Helmholtz, "Handbuch der physiologischen Optik," 3rd Edition, Vol. I, p. 300 *et seq.* *See also* Chap. VI. § 53 footnote.—Trans.

The power is $D=\frac{1}{f}$ where f is expressed in metres. Therefore $D=\frac{1}{0{\cdot}02326}=43\ Dp.$

Ex. 25.

The same as in Ex. 24 except that $r=8$ mm. and $n=\frac{4}{3}$

$$f=\frac{8}{\frac{1}{3}}=24\text{ mm.}\qquad f'=\tfrac{4}{3}\times 24=32\text{ mm.}\qquad D=\tfrac{1000}{24}=41\tfrac{2}{3}\ Dp.$$

Ex. 26.

What is the power and focal distance of a surface of radius r, separating the air on the right from a medium of refractive index n on the left?

In equation (VIII) substituting $n'=1$,

$$f=\frac{nr}{1-n}\qquad f'=\frac{r}{1-n}\qquad D=\frac{1-n}{r}.$$

Ex. 27.

What is the power and focal length of a meniscus lens of thickness d and radii r_1 and r_2, the refractive index being n?

Surface I has power $D_1=\dfrac{n-1}{r_1}$ (Ex. 23).

„ II „ $D_2=\dfrac{1-n}{r_2}$ (Ex. 26).

The vertices S_1 and S_2 (Fig. 28) are the principal points of the surfaces, which in this case are to be considered as the component systems; the reduced distance δ is then $\delta=\frac{d}{n}$. From equation (X) the power D of the meniscus is:

$$D=\frac{n-1}{r_1}+\frac{1-n}{r_2}-\frac{d}{n}\cdot\frac{(n-1)}{r_1}\cdot\frac{(1-n)}{r_2}.$$

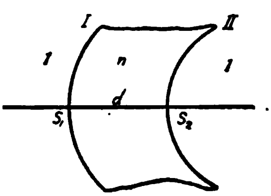

FIG. 28.

Therefore

$$\frac{1}{f}=D=(n-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+\frac{(n-1)^2}{n}\cdot\frac{d}{r_1 r_2}.$$

The focal length $f=\frac{1}{D}$.

Ex. 28.

What is the error involved in the determination of the power of a lens, if in the above the thickness d is neglected?

The power of a very thin lens is $(n-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$, hence from equation (27) the error is $\frac{(n-1)^2}{n}\cdot\frac{d}{r_1 r_2}$

Ex. 29.

What is the amount of this error for an equi-biconvex lens of thickness 5 mm. if $r_1=-r_2=10$ cm. and $n=1{\cdot}5$?

From Ex. 28 the error is

$$-\frac{(\frac{1}{2})^2}{\frac{3}{2}}\times\frac{0{\cdot}005}{(0{\cdot}1)^2}=-\tfrac{1}{12}\ Dp.$$

Ex. 30.

What is the error for a lens having a plane surface?

The expression $\frac{(n-1)^2}{n}.\frac{d}{r_1 r_2}$ disappears if one of the radii becomes infinitely great; hence the error in the expression for the power is zero.

Ex. 31.

What is the power of a **concentric** lens, *i.e.*, one whose centres of the curved surfaces coincide?

In the expression for D (Ex. 27) putting $d=r_1-r_2$, then:

$$D=(n-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+\frac{(n-1)^2}{n}.\frac{(r_1-r_2)}{r_1 r_2}$$

$$=(n-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-\frac{(n-1)^2}{n}.\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

$$=\frac{n-1}{n}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

This value holds when both surfaces lie on the same side of the common centre. If this be not the case the lens will be bi-convex having a thickness r_1+r_2.

Hence putting $-r$ for r_2, the expression for D becomes

$$D=\frac{n-1}{n}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\qquad\ldots\qquad\ldots\ (18a)$$

Ex. 32.

What is the thickness of a concentric lens whose power is $-2Dp.$ and whose radius $r_2=3$ cm. and $n=1{\cdot}5$?

From Ex. 31. $-2=\frac{\frac{3}{2}-1}{\frac{3}{2}}\times\left(\frac{1}{r_1}-\frac{1}{0{\cdot}03}\right)$ from which $\frac{1}{r_1}=27{\cdot}3$

or $r_1=3{\cdot}6$ cm. The thickness is the difference of the radii and is 6 mm. Lenses of this kind would be specially suitable as glasses to correct short sight, as will be seen later, if they were not so heavy.

Ex. 33.

What is the power of a sphere with radius r and refractive index n?

In equation (18a) put the radii equal to each other, then

$$D=\frac{2\,(n-1)}{nr}.$$

Ex. 34.

What is the power of the spherical lens of the eye of a fish, say the carp's, whose $r=2{\cdot}5$ mm. and $n=1{\cdot}68$?

From Ex. 33 we have

$$D = \frac{2 \times 0{\cdot}68}{1{\cdot}68 \times 0{\cdot}0025} = 324\ Dp.$$

Ex. 35.

What is the power of a **Null Lens** (that is a meniscus with equal radii) ?

Ex. 27 gives, substituting $r_1 = r_2 = r$,

$$D = \frac{(n-1)^2}{n} \cdot \frac{d}{r^2}$$

Ex 36.

Where are the principal points of a meniscus whose radii are r_1 and r_2, thickness d, and refractive index n ?

From equation (XI) $h = -\frac{d}{n} \cdot \frac{D_2}{D}$, whilst from Ex. 26, $D_2 = \frac{1-n}{r_2}$, therefore $H = \frac{(n-1)\,d}{n\,r_2 D}$. In the same way $H' = -\frac{d}{n} \cdot \frac{D_1}{D} = -\frac{(n-1)d}{n\,r_1\,D}$

The distances h and h' are measured from the vertices S_1 and S_2 respectively of the lens.

Ex. 37.

What is the power and focal length of a meniscus, and where are the principal points and focal points, if $r_1 = 10$ cm., $r_2 = 20$ cm., $d = 3$ cm., $n = 1{\cdot}5$.

From the formula (Ex. 27) for D the power is

$$\frac{1}{f} = \left(\tfrac{3}{2} - 1\right)\left(\tfrac{1}{10} - \tfrac{1}{20}\right) + \frac{\left(\tfrac{3}{2} - 1\right)^2}{\tfrac{3}{2}} \times \frac{3}{10 \times 20} = \frac{11}{400}$$

Therefore the power is $D = \frac{1100}{400} = 2\tfrac{3}{4}\ Dp.$

The focal length is $f = \frac{400}{11} = 36\tfrac{4}{11}$ cm.

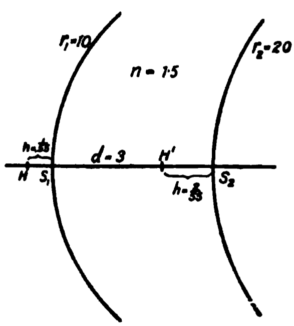

FIG. 29.

From Ex. 36.

$$h' = -\frac{\left(\tfrac{3}{2} - 1\right) \times 3}{\tfrac{3}{2} \times 10 \times \tfrac{11}{4}} = -\tfrac{2}{55}\ \text{m.} \qquad h = \frac{\left(\tfrac{3}{2} - 1\right) \times 3}{\tfrac{3}{2} \times 20 \times \tfrac{11}{4}} = \tfrac{1}{55}\ \text{m.}$$

In Fig. 29, $HS_1 = h$ is measured towards the left, since h is positive, and h', since it is negative is measured also to the left from S_2.

Ex. 38.

What is the power of an equi-bi-convex lens of radius r and thickness d, if $n=\frac{3}{2}$? Also where do the principal points lie?

From Example 27 putting $r_1=r$, $r_2=-r$ and $n=\frac{3}{2}$, we obtain

$$D=\tfrac{1}{2}\left(\frac{1}{r}+\frac{1}{r}\right)-\tfrac{1}{6}\cdot\frac{d}{r^2}=\frac{1}{r}-\tfrac{1}{6}\,\frac{d}{r^2}$$

From Ex. 36

$$h=-\frac{1}{3r}\cdot\frac{d}{D} \qquad h'=-\frac{1}{3r}\cdot\frac{d}{D}$$

Since h and h' are both negative, the principal points are measured from S_1 and S_2 into the interior of the lens.

Ex. 39.

In the above example where do the principal points lie, if the thickness d of the lens is small in proportion to the radius?

Since d is small compared with r, the fraction $\frac{1}{6}\,\frac{d}{r^2}$ may be neglected. Then

$$D=\frac{1}{r} \text{ and } h=-\frac{d}{3} \text{ whilst } h'=-\frac{d}{3}\,.$$

The principal points therefore lie within the lens at points one-third the thickness of the lens from the respective vertices.

Ex. 40.

What is the power of an equi-bi-concave lens and where do the principal points lie?

In the two preceding examples, if we put $-r$ for r, the principal points lie always inside the lens, since in the equations for h and h' (Ex. 38) both r and D change their signs and thus h and h' remain negative.*

Ex. 41.

What is the limiting thickness for an equi-bi-convex lens $\left(n=\frac{3}{2}\right)$ for which the principal points will lie at the vertices of the surfaces?

From Ex. 38.

$$h=h'=-\frac{d}{3rD}=-\frac{d}{3-\dfrac{d}{2r}}$$

When $d<2r$ it will be seen that H lies to the left of H'.

When $d=2r$ (spherical lens) the principal points coincide.

When $d>2r$, the principal points cross over; if they lie at the vertices, then $h=h'=-d$, and we have

$$-d=-\frac{d}{3-\dfrac{d}{2r}} \text{ or } d=4r.$$

* Also, h and h' remain always less than d; that is, the principal points lie always in the interior of the lens.—Trans.

Ex. 42.

What is the power of the crystalline lens of the human eye, assuming the constants given previously, *viz.*, in Fig. 27, $r_2=10$ mm., $r_3=-6$ mm., N is the refractive index of the lens material=1·43. The lens is situated in a medium of refractive index n=1·3365, whilst the thickness d of the lens is 3·6 mm.

From equation (VIII) the powers D_1 and D_2 of the two surfaces are

$$D_1 = \frac{1\cdot 43 - 1\cdot 3365}{0\cdot 01} = \frac{0\cdot 0935}{0\cdot 01} = 9\cdot 35\ Dp.$$

$$D_2 = \frac{1\cdot 3365 - 1\cdot 43}{-0\cdot 006} = 15\cdot 58\ Dp.$$

The reduced distance between the surfaces is

$$\delta = \frac{d}{N} = \frac{0\cdot 0036}{1\cdot 43} = 0\cdot 002518 \text{ m.}$$

Hence from (X)

$$D = 9\cdot 35 + 15\cdot 58 - 9\cdot 35 \times 15\cdot 58 \times 0\cdot 002518 = 24\cdot 55\ Dp.$$

NOTE.—Had the thickness of the lens been neglected, the power would have been 24·93 Dp, which differs from the above by only 0·38 Dp.

Ex. 43.

Where are the principal points of the crystalline lens of the human eye?

From equation (XI)

$$H = -\frac{D_2\,\delta}{D} = \frac{-15\cdot 58 \times 0\cdot 002518}{24\cdot 55}\,m = -1\cdot 598 \text{ mm}$$

$$H' = -\frac{D_1\,\delta}{D} = \frac{-9\cdot 35 \times 0\cdot 002518}{24\cdot 55} = -0\cdot 9578 \text{ mm.}$$

Hence

$$h = nH = -1\cdot 3365 \times 1\cdot 598 = -2\cdot 135 \text{ mm.}$$

$$h' = nH' = -1\cdot 3365 \times 0\cdot 9578 = -1\cdot 282 \text{ mm.}$$

In Fig. 30—$H_0\,S_2 = 2\cdot 135$ mm. $H_0'\,S_3 = 1\cdot 282$ mm.

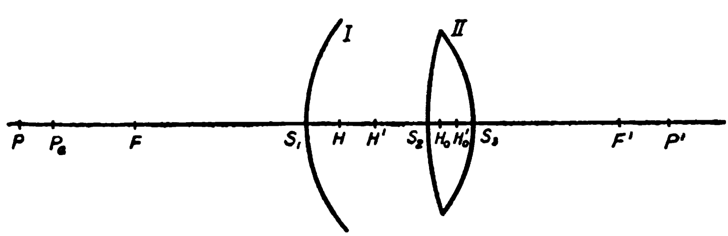

FIG. 30.

Ex. 44.

What is the power of the *whole* schematic eye? What are the focal lengths and where do the principal points lie? Depth of anterior chamber is 3·6 mm.

The eye consists of a combination of the two systems :—

I. The cornea of power D_1=43 Dp. (Ex 24.)
II. The crystalline lens of power D_2=24·55 Dp. (Ex. 42.)

The distance between the adjacent principal points is
$d = S_1 H_0 = S_1 S_2 + S_2 H_0 = 3 \cdot 6 + 2 \cdot 135 = 5 \cdot 735 \text{ mm} = 0 \cdot 005735 \text{ m}.$

Reducing we obtain $\delta = \dfrac{0 \cdot 005735}{1 \cdot 3365} = 0 \cdot 00429$ m.

Hence the power of the whole eye is :—

$$D = 43 + 24 \cdot 55 - 0 \cdot 00429 \times 43 \times 24 \cdot 55 = 63\, Dp.$$

Further from (XI)

$$H = h = -\frac{D_2 \delta}{D} = -\frac{24 \cdot 55 \times 4 \cdot 29}{63} = -1 \cdot 67 \text{ mm}.$$

$$H' = \frac{h'}{1 \cdot 3365} = -\frac{D_1 \delta}{D} = -\frac{43 \times 4 \cdot 29}{63} = -2 \cdot 927 \text{ mm}.$$

i.e.,

$$h' = -2 \cdot 927 \times 1 \cdot 3365 = -3 \cdot 912 \text{ mm}.$$

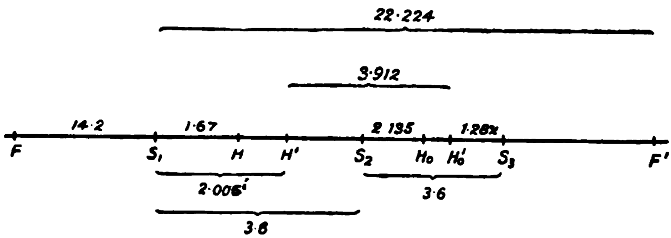

FIG. 31.

In the above h is the distance of the first principal point of the whole eye from the vertex S_1 of the cornea. In Fig. 31, if H is the first principal point of the whole eye, then $-h = S_1H = 1{\cdot}67$ mm. The negative sign means that this point lies to the right of the vertex of the cornea, that is inside the eye. *The proximity of the first principal point to the pole of the cornea makes it specially suitable as a point of reference for axial distances.*

h' is the distance of the second principal point of the whole eye from the second principal point of the crystalline lens. If in Fig. 31 H′ is the second principal point of the whole eye, then

$$-h' = H' H_0' = 3 \cdot 912 \text{ mm}.$$

The negative sign of h' means that it is reckoned to the left of H_0'. Hence the distance of the second principal point H' of the whole eye from the vertex S_1 of the cornea (Fig. 31) is :

$S_1 H' = S_1 S_3 - S_3 H_0' - H' H'_0 = 7 \cdot 2 - 1 \cdot 282 - 3 \cdot 912 = 2 \cdot 006$ mm.

Since the first medium is air, the first focal distance f of the eye is the reciprocal of the power, that is,

$$f = \frac{1}{63} \text{ m} = 15 \cdot 87 \text{ mm}.$$

Since the second focal distance f' lies in a medium of refractive index 1·3365

$$f = \frac{1 \cdot 3365}{63} \text{ m} = 21 \cdot 22 \text{ mm}.$$

Ex. 45.

Where do the focal points of the whole eye lie ?

Since the focal distance f and f' are positive (Ex. 44), f is measured to the left of the first principal point H, and f' to the right of H'. In order to find the distance of the first focal point from the

vertex of the cornea, the distance $S_1H = 1{\cdot}67$ must be subtracted from the focal distance $f = 15{\cdot}87$ or

$$FS_1 = 14{\cdot}2 \text{ mm. (Fig. 31).}$$

To find the distance of the second focal point F' from the vertex of the cornea, we must add the second focal distance $21{\cdot}22$ to the distance $H'S_1 = 2{\cdot}006$, that is

$$F'S_1 = 23{\cdot}226 \text{ mm.}$$

Ex. 46.

What is the distance between two systems of powers D_1 and D_2, so that the resultant power is zero?

From equation (X) for $D = 0$ the reduced distance between the adjacent principal points is

$$\delta = \frac{1}{D_1} + \frac{1}{D_2}.$$

If the two systems be situated in air, the *separation* δ becomes equal to the sum of the focal lengths (telescopic system).

34. Focal Point Vergence.

Vergences may be reckoned from any point on the axis, as for example, from the vertices of the system, from the focal points, the nodal points, and so on. The term vergence, when used by itself, will be understood to signify the vergence referred to the principal points (after Gullstrand). Otherwise we shall speak of vergence referred to the focal points, vergence from the nodal points, &c. For a system referred to the focal points, the introduction of reduced vergences scarcely simplifies the various formulæ involved.

Putting $X = \frac{n}{x}$, $X' = \frac{n'}{x'}$, equations (14) and (13) §16 and (16*a*) §19 become:

$$XX' = D^2, \quad \beta = \frac{X}{D} = \frac{D}{X'}, \quad \gamma = \frac{n}{n'} \cdot \frac{D}{X} = \frac{n}{n'} \cdot \frac{X'}{D} \qquad (19)$$

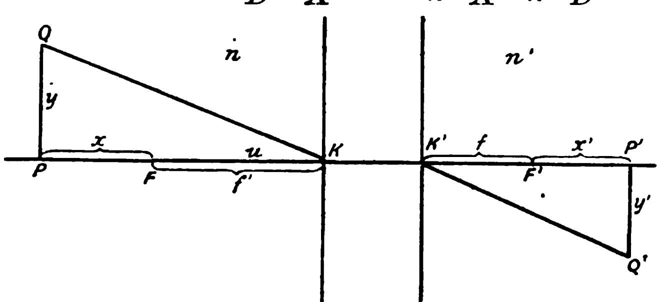

FIG. 32.

35. Nodal Point Vergence.

If b and b' be the distances of two conjugate points from the nodal points K and K', we have from § 23, and referring to Fig. 32:

$$x = b - f', \quad x' = b' - f \quad \text{where } b = KP \text{ and } b' = K'P' \qquad (20)$$

Further

$$xx' = (b - f')\ (b' - f)$$

or

$$\frac{f'}{b} + \frac{f}{b'} = 1 \quad \ldots \quad \ldots \quad (21)$$

From the right angled triangles KQP and $K'Q'P'$ it follows :

$$\left.\begin{aligned} \beta &= \frac{y'}{y} = \frac{b'}{b} \text{ and since } \gamma = \frac{1}{\beta} \cdot \frac{n}{n'} \\ \therefore\ \gamma &= \frac{u'}{u} = \frac{b}{b'} \cdot \frac{n}{n'} \end{aligned}\right\} \quad \ldots \quad (22)$$

In order to realise fully the symmetry of the formulæ for a system referred to the nodal points, put

$$f' = \phi \text{ and } f = \phi' \quad \ldots \quad \ldots \quad (23)$$

Then, from the definition :—

In a nodal point system, the first focal distance ϕ is equal to the distance of the first nodal point from the first focal point, the second focal distance ϕ' is the distance of the second nodal point from the second focal point,

$$\frac{\phi}{\phi'} = \frac{n'}{n} \quad \ldots \quad \ldots \quad (24)$$

In a nodal point system we reduce distances by *multiplying* by the corresponding refractive index.

Then for the reduced nodal point vergences we have :

$$B = \frac{1}{nb} \qquad B' = \frac{1}{n'b'}$$

From equation (24) we obtain :

$$\frac{1}{n\phi} = \frac{1}{n'\phi'} = D_K \quad \ldots \quad \ldots \quad (24a)$$

where D_K is the **Nodal Point Power,**

and equation (21) becomes

$$B + B' = D_K \quad \ldots \quad \ldots \quad (25)$$

from which

$$\beta = \frac{n}{n'} \cdot \frac{B}{B'} \text{ and } \gamma = \frac{B'}{B} \quad \ldots \quad \ldots \quad (26)$$

Moreover, from equations (23) and (24a) it follows, since $\frac{n}{n} = \frac{f}{f'}$

$$D = D_K nn' \quad \ldots \quad \ldots \quad (27)$$

With the new nomenclature, equation (24) §24 becomes

$$\phi' = \frac{\phi_1'\phi_2'}{k - \phi_2 - \phi_1'}$$

Let n, N, n' be the three refractive indices of the media under consideration, then :

$$\frac{1}{n\phi_1} = \frac{1}{N\phi_1'} = D_{1K}$$

$$\frac{1}{N\phi_2} = \frac{1}{n'\phi_2'} = D_{2K}$$

and $kN = \chi$.

Hence, if D_K is the power of the whole system

$$D_K = D_{1K} + D_{2K} - \chi D_{1K} D_{2K} \quad \ldots \quad (28)$$

an equation which conforms strictly with equation (16) §32. If a system be in air, $n' = n = 1$, the two focal distances become equal and the principal and nodal points coincide ; then $D = D_K$.

The nodal point system and the principal point system are, in principle, of equal value. Both systems are, in a way, artificial, in that the simplicity of their notation is a result of the ordinary law of refraction, and disappears when the light is refracted according to some other law, as in double refracting media ; whilst the system of equations referred to the focal points holds generally even in the latter case.

Ex. 47.

Where are the nodal points of a single refracting surface ?

The nodal points are defined as points such that for a ray passing through the first and subsequently through the second, the convergence ratio is unity ; that is, the incident and emergent rays make equal angles with the axis. Since a ray directed towards the centre of a sphere passes through the surface unrefracted, the nodal points must coincide at the centre of the sphere.

Ex. 48.

What is the value of D_K for a surface of radius r separating two media of refractive indices n and n' ?

Since $D_K = \dfrac{D}{nn'}$, then $D_K = \dfrac{n' - n}{nn'r}$.

Ex. 49.

What is the power of a sphere in air. of radius r and refractive index n ?

A sphere consists of a combination of two surfaces with equal and opposite radii, whose nodal points coincide. Hence in equation (28) $\chi = o$ and we obtain

$$D_K = D = D_{1K} + D_{2K}$$

Now

$$D_{1K} = \frac{n-1}{nr} \text{ and } D_{3K} = -\frac{1-n}{nr}$$

Hence

$$D = \frac{2(n-1)}{nr}$$

in agreement with Ex. 33.

Ex. 50.

What is the power of a sphere in air, of refractive index n and radius R, which encloses symmetrically a small sphere of refractive index N and radius r?

Since the nodal points of all the surfaces as well as those of the whole system lie at the centre, we conclude from equation (28) that in such a case the power of the system must be equal to the sum of the nodal point powers.

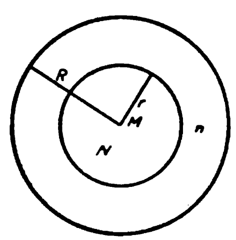

FIG. 33.

Then (*see* Fig. 33),

$$D = 2\left(\frac{n-1}{nR} + \frac{N-n}{nNr}\right).$$

Suppose

$$n = \tfrac{4}{3},\ N = \tfrac{3}{2},\ R = 2 \text{ and } r = 1.$$

Then

$$D = 2\left(\frac{\frac{1}{3}}{\frac{4}{3} \times 2} + \frac{\frac{3}{2} - \frac{4}{3}}{\frac{3}{2} \times \frac{4}{3}}\right) = \tfrac{5}{12}\ Dp.$$

A sphere of radius R and refractive index N has the power

$$\frac{2(N-1)}{NR} = \frac{2(\frac{3}{2} - 1)}{\frac{3}{2} \times 2} = \tfrac{1}{3}\ Dp.$$

Thus it will be seen that there is a reduction in power when the whole sphere of radius R is filled with a medium of higher refractive index N. This becomes evident, if we consider the two halves of the outer sphere, to the left and right of the enclosed sphere (Fig. 33) as acting like diverging lenses, which become stronger the higher is their refractive index. If they both have a refractive index of N the whole system becomes a homogeneous sphere of refractive index N and the parts under consideration produce a stronger diverging action than before. An analogous case exists when a lens is separated into layers of refractive index increasing towards the centre, as is the case of the lenses of the eyes of nearly all the higher animals, especially in human beings.

36. The Curvature of the Image.

If an object or an image be not formed in a plane surface, but lies on a spherical surface, the reciprocal value of the radius of the surface is called the *curvature*.

If the object have curvature $\frac{1}{R}$, then $\frac{n}{R}$ is the reduced curvature of the object where n is the refractive index in the object space. Similarly $\frac{n'}{R'}$ is the reduced curvature of the image.

On condition that the rays are not made to pass through a very narrow diaphragm, we have the law (Petzval Law*) :—

The difference between the reduced curvatures before and after refraction through a centred system of spherical surfaces is equal to the sum of the nodal point powers of the individual refracting surfaces.

$$\frac{n}{R} - \frac{n'}{R'} = D_{1K} + D_{2K} + D_{3K} + \ . \ . \ .$$

Ex. 51.

Assuming the data for the schematic eye (§33), what is the curvature of the image formed by the human eye?

The powers of the three surfaces in question are, say, D_1, D_2, D_3; the values of D_K are obtained from these values by dividing by the product of the refractive indices before and behind the respective surfaces. Then, from Exs. 42 and 43;

$$\text{Cornea, } D_1 = 43\ Dp.;\ D_{1K} = \frac{43}{1 \cdot 3365} = 32 \cdot 17\ Dp.$$

$$\text{1st surface of lens, } D_2 = 9 \cdot 35;\ D_{2K} = \frac{9 \cdot 35}{1 \cdot 3365 \times 1 \cdot 43} = 4 \cdot 89\ Dp.$$

$$\text{2nd ,, ,, ,, } D_3 = 15 \cdot 58;\ D_{3K} = \frac{15 \cdot 58}{1 \cdot 3365 \times 1 \cdot 43} = 8 \cdot 15\ Dp.$$

$$D_{1K} + D_{2K} + D_{3K} = 45 \cdot 22\ Dp.$$

Remembering that the refractive index of the object space is unity, and of the image space 1·3365, we have

$$\frac{1}{R} - \frac{1 \cdot 3365}{R'} = 45 \cdot 22$$

If the object be plane, so that $R = \infty$ then

$$-R' = \frac{1 \cdot 3365}{45 \cdot 22} = 0 \cdot 029 \text{ m} = 29 \text{ mm}.$$

Hence the image of a plane object is formed at the retina, on a sphere of radius 29 mm., the concave side of which is turned towards the object space (since R' has a negative value). Since no account has been taken of the structure of the lens of the eye, this value is only approximate.

* *See* Gleichen: Lehrbuch der geometrischen Optik. Leipzig, 1902, p. 218, for a proof of this rule.

Ex. 52.

What is the curvature of the image produced by a thin spectacle lens, if r_1 and r_2 be the radii and n the refractive index?

Let D_1 and D_2 be the powers of the first and second surfaces, then:

$$D_1=\frac{n-1}{r_1} \qquad D_{1K}=\frac{n-1}{nr_1}$$

$$D_2=\frac{1-n}{r_2} \qquad D_{2K}=\frac{1-n}{nr_2}$$

Now for a thin lens,

$$\frac{1}{f}=D_1+D_2=(n-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

The curvature of the image of a plane object for a thin lens,

$$\frac{1}{R'}=D_{1K}+D_{2K}=\frac{n-1}{n}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

Hence

$$\frac{n}{R'}=\frac{1}{f} \text{ or } R'=nf.$$

The image of a plane object is formed by a spectacle lens on a sphere whose radius is equal to the product of the focal length and the refractive index.

The curvature is thus wholly independent of the shape of the glass (meniscus, convex, &c.).

Translators' Appendix to Chapter III.

In Chapter II. lenses were classified broadly into two main groups—**Convergent** and **Divergent**. The former are thickest along the optical axis, and the latter thinnest along the optical axis; the former are sometimes called *Positive* and the latter *Negative*. These properties depend, however, on the thickness of the lens, and these statements are strictly true only if the lens be thin. Whilst most lenses used in practice are thin compared with the radii of their bounding surfaces, so that this classification is correct, there are other lenses for which the classification needs qualifying.

In what follows we assume the lenses to be made of glass whose $n > 1$; the lenses being surrounded by air ($n = 1$).

Consider the general case of a simple lens of refractive index n, radii of curvature r_1 and r_2, thickness d, with a medium in front of refractive index n_1 and one behind of refractive index n_2.

Then

$$\text{Power of surface I} = D_1 = \frac{n-n_1}{r_1}$$

$$\text{,, \quad ,, \quad ,, \quad II} = D_2 = \frac{n_2-n}{r_2}$$

$$\begin{aligned}\text{Power of combination} &= \text{Power of lens}\\ &= D\\ &= D_1+D_2-\delta\cdot D_1 D_2\\ &=\frac{n-n_1}{r_1}+\frac{n_2-n}{r_2}-\frac{d}{n}\cdot\frac{(n-n_1)(n_2-n)}{r_1 r_2}\end{aligned}$$

If, as is usual, the lens be situated in air, then

$$n_1 = n_2 = 1.$$

and

$$D = \frac{n-1}{r_1} + \frac{1-n}{r_2} - \frac{d}{n} \cdot \frac{(n-1)(1-n)}{r_1 r_2}$$

$$= (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{(n-1)^2}{n} \cdot \frac{d}{r_1 r_2} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$= \text{power of any lens in air.}$$

In a slightly modified form

$$D = \frac{(n-1)\{(n-1)d - n(r_1 - r_2)\}}{n r_1 r_2} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

The positions of the principal points relative to the vertices of the surfaces are given by

$$\left.\begin{aligned} h = H = -\frac{D_2}{D}\delta = -\frac{(1-n)\delta}{r_2 D} = \frac{(n-1)d}{nDr_2} \\ h' = H' = -\frac{D_1}{D}\delta = -\frac{(n-1)\delta}{r_1 D} = -\frac{(n-1)d}{nDr_1} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (3)$$

It is clear from (2) that the sign of the power D depends upon the signs of r_1 and r_2; but it also depends on the value of d. Taking the case of a bi-convex lens (r_1 positive, r_2 negative), and expressing the *numerical* values of r_1 and r_2 by R_1 and R_2, so that

$$r_1 = R_1\,; \quad -r_2 = R_2$$

we have

$$D = \frac{-(n-1)\ \{(n-1)d - n(R_1 + R_2)\}}{nR_1R_2}$$

Now this is positive only so long as

$$(n-1)d - n(R_1 + R_2) \text{ is negative.}$$

i.e.

$$d < \frac{n(R_1 + R_2)}{n-1}.$$

Usually this is the case, so that a bi-convex lens is *usually* positive or convergent. But if $d = \frac{n(R_1 + R_2)}{n-1}$, the power is zero, and the lens is said to be *Telescopic*. And if $d > \frac{n(R_1 + R_2)}{n-1}$, we have the case of a *divergent* bi-convex lens.

Below is a table showing some typical lenses, giving their powers, positions of principal points, etc.—Trans.

TABLE I.—VARIOUS FORMS OF LENSES.

Kind of Lens.	R_1 and R_2 are the numerical values of the radii.	Power in Air $= D$; d = thickness of lens.	Distances of principal points from vertices of surfaces.	—	—
Bi-convex in general.	$r_1 = R_1 > 0$ $-r_2 = R_2 > 0$	$(n-1)\left(\frac{1}{R_1}+\frac{1}{R_2}\right) - \frac{(n-1)^2 d}{nR_1R_2}$	$h = -\frac{(n-1)d}{nDR_2}$ $h' = -\frac{(n-1)d}{nDR_1}$	Convergent Telescopic or Divergent.	
Equi-Bi-convex.	$r_1 = -r_2 = R > 0$	$\frac{2(n-1)}{R} - \frac{(n-1)^2 d}{n.R^2}$	$h = h' = -\frac{Rd}{2Rn-(n-1)d}$	Convergent Telescopic or Divergent.	Principal points lie anywhere on optical axis depending on value of thickness, d.
Concentric (positive).	$d = R_1 + R_2$ $= -(h+h')$	$\frac{n-1}{n}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$	$h = -R_1$ $h' = -R_2$	Convergent always.	Principal points lie at common centre of lens surfaces.
Spherical	$r_1 = -r_2 = R > 0$	$\frac{2(n-1)}{nR} = \frac{4(n-1)}{nd}$	$h = h' = -R$	Convergent always.	Principal points coincide at common centre of lens surfaces.
Telescopic	$d = \frac{n}{n-1}(R_1+R_2)$	$D = 0$	At Infinity	Telescopic	If $n = \frac{3}{2}$ and $R_1 = R_2 = R$, then $d = 6R$.
Plano-Convex.	$r_2 = \infty$ $r_1 = r = R$	$\frac{n-1}{R}$	$h = 0$ $h' = -\frac{d}{n}$	Convergent always.	First principal point on curved surface.

Positive meniscus.	$r_1 = R_1 > 0$ $r_2 = R_2 > R_1$	$(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{(n-1)^2 d}{nR_1R_2}$	$h = \frac{(n-1)d}{n\,DR_2}$ $h' = -\frac{(n-1)d}{n\,DR_1}$	Convergent always.	
Null lens	$r_1 = r_2 = r$ $= R > 0$	$\frac{(n-1)^2 d}{nR^2}$	$h = -h'$ $= \frac{R}{n-1}$	Convergent always.	Sometimes called Lens of Zero curvature.
Bi-concave in general.	$-r_1 = R_1 > 0$ $r_2 = R_2 > 0$	$-\left\{(n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \frac{(n-1)^2 d}{nR_1R_2}\right\}$	$h = \frac{(n-1)d}{n\,DR_2}$ $h' = \frac{(n-1)d}{n\,DR_1}$	Divergent always.	Principal Points lie always in interior of lens.
Equi-Bi-concave.	$-r_1 = r_2$ $= R_1 = R_2 = R > 0$	$-\left\{\frac{2(n-1)}{R} + \frac{(n-1)^2 d}{nR^2}\right\}$	$h = h'$ $= -\frac{Rd}{2Rn + (n-1)d}$	Divergent always.	Principal points lie always in interior of lens.
Plano-concave.	$r_2 = \infty$ $-r_1 = -r = R > 0$	$-\frac{(n-1)}{R}$	$h = o$ $h' = -\frac{d}{n}$	Divergent always.	First principal point on curved surface.
Negative meniscus.	$r_1 = R_1 > 0$ $r_2 = R_2 < R_1$	$(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{(n-1)^2 d}{nR_1R_2}$	$h = \frac{(n-1)d}{n\,DR_2}$ $h' = -\frac{(n-1)d}{n\,DR_1}$	Divergent Telescopic or Convergent.	
Concentric (negative).	$d = R_1 - R_2$ $= -(h + h')$	$\frac{n-1}{n}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$	$h = -R_1$ $h' = +R_2$	Divergent always.	Principal points coincide at common centre of lens surfaces.

CHAPTER IV.

Limitation of Rays. Sharp and Blurred Image Formation. Depth of Focus.

37. Limitation of Rays.

The rays taking part in image formation are "regulated" or limited by diaphragms, lens holders, &c. In what follows only rays travelling near to the axis will be considered. In Fig. 34, $\alpha\beta$ is any circular diaphragm whose centre E lies on the optical axis. This diaphragm may be placed in front of, behind, or inside a centred optical system. It limits the rays passing through the system, and is therefore called the **Opening** or **Aperture diaphragm**.

Of the rays from the axial point P, only those can traverse the system which pass through the aperture diaphragm. Although in Fig. 34 the diaphragm is placed between two convex lenses I and II, the following considerations are quite general.

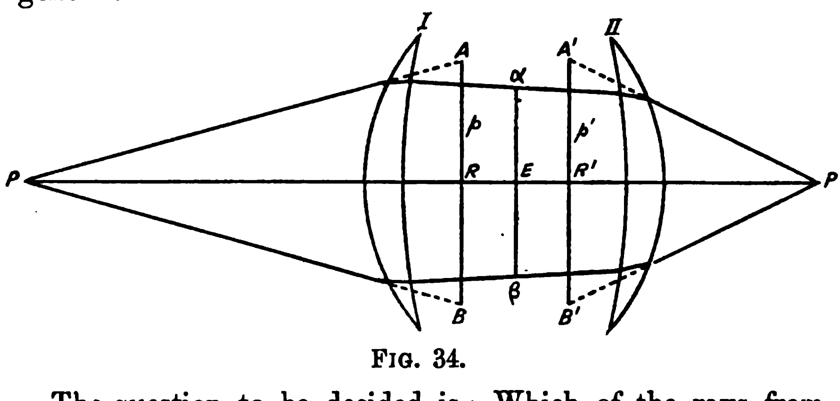

FIG. 34.

The question to be decided is: Which of the rays from the point P in the object space actually pass through the diaphragm and take part in the production of the image P'? To answer this question, consider the image of the diaphragm $\alpha\beta$ formed to the left by system I. Let the image be AB. In the case shown in the figure, it is virtual, magnified, and erect. Now the point α is conjugate to A with reference to system I, and β is conjugate to B. Consequently all rays passing through A in the object space, after refraction through system I, pass through the point α. If therefore PA be drawn, this ray after refraction through system I, must pass through the upper part α on the edge of the

diaphragm. Similarly a ray through B must pass through the lower part β of the diaphragm after refraction through system I. In this way it is apparent that the circular space with diameter AB, which is called the **Entrance Pupil,** determines those rays* in the object space which really traverse the system.

Consider now the image of the diaphragm $\alpha\beta$ formed on the right by system II. In this way the exit pupil is obtained, viz., the circular space of diameter $A'B'$, which determines the cone of rays in the image space; the rays, after filling and traversing the diaphragm $\alpha\beta$, converge on the image point P' as a cone of rays which appears to fill and emerge from the exit pupil. It is easily seen that AB

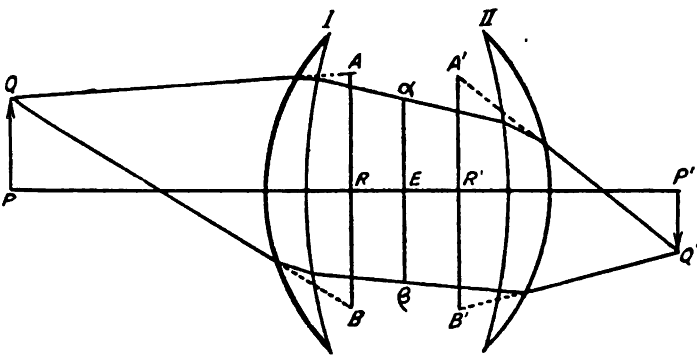

FIG. 35.

and $A'B'$ are conjugate with reference to the whole system I and II. The image $A'B'$ of the diaphragm is called the **Exit Pupil.** In Fig. 34 the actual path of the rays is indicated by full lines. The entrance and exit pupils determine not only those rays from a point on the axis, but also those rays from *any* point in the object space. This is shown in Fig. 35. There, it will be observed, the rays from a point Q on one side of the axis first pass through the entrance pupil AB, then through the diaphragm $\alpha\beta$ and finally converge on the image point Q' after virtually filling and traversing the exit pupil $A'B'$. As before, the reason for this procedure lies in the fact that A and α with reference to system I, and α and A' with reference to system II are conjugate. Hence a ray in the object space, traversing the system in such a direction as to pass through A, after refraction through system I, passes through the point α and finally leaves the system II as if it came from A'.

* *i.e.*, the *effective* or operative rays.—Trans.

38. Definition of Aperture Diaphragm.

If there be several diaphragms (to which in certain circumstances lens holders belong) in an optical system, then the **aperture diaphragm** corresponding to an object point on the axis is defined in the following way: Find the images in the object space of all the diaphragms by the portion of the system lying before them; then the **effective aperture** is that one whose diaphragm image subtends the smallest angle at P.

39. Entrance and Exit Pupils of Centred Systems.

Fig. 36 shows a centred system with principal points H and H' and principal foci F and F', so that

$$FH=f \text{ and } F'H'=f'$$

are respectively the first and second focal distances of the system.

AR represents one-half of the entrance pupil. It is required to find the image of AR formed by the whole system. From A draw a ray parallel to the axis cutting the

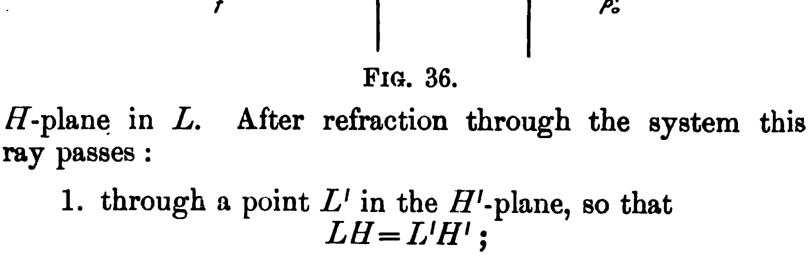

FIG. 36.

H-plane in L. After refraction through the system this ray passes:

1. through a point L' in the H'-plane, so that
$$LH=L'H';$$

and 2. through the second focal point F', since it is parallel to the axis in the object space.

Consider a second ray passing through A, namely, that from the first focal point F, cutting the H-plane in J. After refraction through the system this ray must travel

1. parallel to the axis

and 2. through a point J' in the H'-plane, so that

$$JH=J'H'.$$

The ray through the points F' and L' and the ray through J' parallel to the axis intersect in A' which is conjugate to A. If we drop the perpendicular $A'R'$ from A' on to the axis, then $A'R'$ is the image of AR, that is the radius of the exit pupil.

Now let

$$AR=\rho \qquad A'R'=\rho'$$

and further

$$FR=p_0 \qquad F'R'=p_0'$$

Then from the similar triangles JHF and ARF,

$$\frac{JH}{AR}=\frac{FH}{FR}$$

that is

$$\frac{\rho'}{\rho}=\frac{f}{p_0} \qquad \dots \qquad \dots \qquad (1)$$

From the similar triangles $A'R'F'$ and $L'H'F'$ we obtain in the same way :

$$\frac{\rho'}{\rho}=\frac{p_0'}{f'} \qquad \dots \qquad \dots \qquad (2)$$

In Fig. 36 let P and P' be any two conjugate points and let

$$RP=p \qquad R'P'=p'$$

then, as in equation (14) §16, viz. $xx'=ff'$, since

$$x=p-p_0 \qquad x'=p'-p_0'$$

we obtain

$$(p-p_0)\,(p'-p_0')=ff'.$$

Substituting the values found in (1) and (2), this latter equation becomes

$$f\cdot\frac{\rho}{\rho'}\cdot\frac{1}{p}+f'\cdot\frac{\rho'}{\rho}\cdot\frac{1}{p'}=1 \qquad \dots \qquad (3)$$

From equation (11a) §16 the magnification $\frac{y'}{y}$ is given by

$$\beta=\frac{y'}{y}=\frac{f}{x}=\frac{f}{p-f\cdot\frac{\rho}{\rho'}}=\frac{f}{p}\cdot\frac{1}{1-f\cdot\frac{\rho}{\rho'}\cdot\frac{1}{p}}$$

which, on transposing from (3), takes the form

$$\beta=\frac{p'}{p}\cdot\frac{\rho}{\rho'}\cdot\frac{f}{f'}=\frac{p'}{p}\cdot\frac{\rho}{\rho'}\cdot\frac{n}{n'} \qquad \dots \qquad \dots \qquad (4)$$

40. Complete Vergences.

As we have seen in § 31, the vergence of a beam of rays is inversely proportional to the distance of the object point or image point from the corresponding point of reference; the centres of the entrance pupil and exit pupil are these points of reference in the present case. But it is evident that the vergence increases also as the entrance pupil and exit pupil increase, and indeed may be considered proportional to the areas of the latter and therefore to the squares of the radii of the two pupils. In contradistinction to the Gullstrand convergences, P and P', which are referred simply to the centres of the entrance pupil and exit pupil, a more complete definition for vergence will now be introduced. Thus, the **Complete Reduced Vergences,** Q and Q', are defined as:

$$Q = \frac{n\rho^2}{p} \qquad Q' = \frac{n'\rho'^2}{p'} \quad \ldots \quad \ldots \quad (5)$$

or

$$Q = \rho^2 P \qquad Q' = \rho'^2 P' \quad \ldots \quad \ldots \quad (5a)$$

where

$$P = \frac{n}{p} \text{ and } P' = \frac{n'}{p'} \quad \ldots \quad \ldots \quad (5b)$$

are the convergences in the usual (Gullstrand's) sense. Employing these quantities in equation (3) § 39, we obtain

$$Q + Q' = D\rho\rho' = \Delta \quad \ldots \quad \ldots \quad (6)$$

where $D = \frac{n}{f} = \frac{n'}{f'}$, is the power in the usual sense and

$$\Delta = D\rho\rho' \,* \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

Δ will be called the **complete power.** Further, equation (4) becomes

$$\beta = \frac{y'}{y} = \frac{Q}{Q'} \cdot \frac{\rho'}{\rho} \quad \ldots \quad \ldots \quad (7a)$$

41. Blurred Image Formation.

In Fig. 37 let R' be the mid-point of the exit pupil of a centred system; $R'K = \rho'$ is the radius of the exit pupil; P and P_1 are two axial object points whose conjugate image points are P' and P_1'. If a screen be placed at P' perpendicular to the axis, a plane object perpendicular to the axis at P

* *See* Der Mechaniker (Berlin-Nicolassee), 1909. Heft, 14–20.

will form on it a sharp image. All other object points will give blurred images and in place of sharp image points, **blur circles** appear, which depend on the size of the exit pupil. For example, an extreme ray from P_1 intersecting the edge of the exit pupil in K, travels towards $P_1{}'$ and cuts the screen

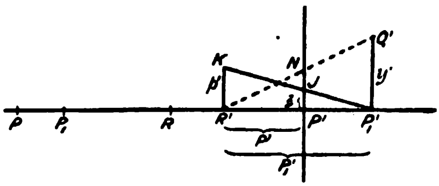

FIG. 37.

in J. Then $JP' = z$ is the radius of the blur circle corresponding to the point P_1. From the similar triangles $KR'P_1{}'$ and $JP'P_1{}'$, if $R'P_1{}' = p_1{}'$ and $R'P' = p'$, then

$$\frac{\rho'}{z} = \frac{p_1{}'}{p_1{}' - p'}$$

or

$$z = \rho'\left(1 - \frac{p'}{p_1{}'}\right) \quad \dots \quad \dots \quad (8)$$

From equation (6)

$$Q_1 + Q_1{}' = \Delta$$

hence equation (8) becomes

$$z = \rho' . \frac{Q_1 - Q}{\Delta - Q} \quad \dots \quad \dots \quad (9)$$

where

$$Q = \frac{n\rho^2}{p} \text{ and } Q_1 = \frac{n\rho^2}{p_1} \text{ if } p = PR \text{ and } p_1 = P_1R.$$

Further, let y' in Fig. 37 be the sharp image of an object y at P_1; if $P'N = k$, we have from the similar triangles $R'Q'P_1{}'$ and $R'NP'$:

$$\frac{k}{y'} = \frac{p'}{p_1{}'}$$

or

$$k = \frac{y'}{y} . \frac{p'}{p_1{}'} . y$$

whilst from equation (7a)

$$k = \frac{Q_1}{\Delta - Q} . \frac{\rho'}{\rho} . y \quad \dots \quad \dots \quad (10)$$

From equations (9) and (10) the complete sharp and blurred image formation in the P'-plane is determined. Equation (9) gives the size of the blur circles, and equation (10) the size of the blurred image corresponding to an object y at the distance p_1 (vergence Q_1) from the centre of the exit pupil.

42. Depth of Focus.

The human eye, receiving an image from an optical instrument, does not always experience indistinctness arising from the circles of confusion. Blur circles whose radii are below a certain value z_0 are conceived as sharp image points. Consequently we are not restricted merely to a plane but to a portion of space for the formation of sharp images—a phenomenon which is termed **Depth of Focus.*** The value z_0 may be positive or negative according as the geometrically sharp image point P_1' (Fig. 37) lies behind or in front of the P' plane, measured in the direction of the incident light. If the system be focussed geometrically sharp for a point of complete vergence Q, then on account of depth of focus, those object planes also appear sharp whose complete vergences lie within the value Q_1 reckoned from equation (9); thus, putting $z = \pm z_0$.

$$Q_1 = Q \pm \frac{z_0}{\rho}(\Delta - Q). \quad \ldots \quad \ldots \quad (11)$$

* *See also* Arts. 66 and 137 and onwards for further discussion on depth of focus.—Trans.

CHAPTER V.

Rays Outside the Paraxial Region. Aberrations.

Translators' Introduction.

Aberrations of the Image.

In the preceding chapters we have dealt with the laws and equations governing the position, magnification, &c. of the image of a given object formed by any system of centred spherical refracting surfaces. We have made no mention, however, of the *quality* of the image produced. In practice the image suffers from certain defects or **aberrations** which it is necessary to correct.

In the first place there are defects due to the phenomenon of dispersion arising from the nature of the material of which the optical parts are made. These are called **Chromatic Aberrations.** As in the case of the prism dealt with in Chapter I, an incident beam of white light traversing a lens gives rise to a number of coloured emergent beams all refracted to different extents. The image, consequently, is no longer a distinct image, but consists of a confusion of coloured images. This defect is corrected, as in the case of the achromatic prism, by combining lenses of different kinds of glass.*

Moreover, we have up to the present considered only those rays which lie within a narrow cylindrical space around the optical axis of the system; that is, within the paraxial region. There are reasons however, in practice, why such narrow beams cannot be used. Thus, the laws of physical optics introduce important modifications, for they show that a very narrow beam from a point object cannot produce a point image. Instead, it produces a circular diffraction pattern.† Further, apart from this consideration, the resulting image with such a narrow beam would be so faint as to be useless, whether in instruments for photographic or visual purposes.

Hence in practical optics we must use rays traversing the system which are outside the paraxial region, and in so doing, the *ideal* image which, according to geometrical optics is formed by the paraxial rays, becomes one which possesses certain defects arising from the spherical character of the refracting surfaces.

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* For the process of achromatising *see*
Hovestadt "Jena Glass and its Applications," 1902.
Southall "Principles and Methods of Geometrical Optics." Macmillan, 1918.
—Trans.

† For further consideration of diffraction phenomena, *see* any book on Physical Optics.—Trans.

Of these aberrations, *i.e.*, the *aberrations of sphericity*, we may enumerate five; **Axial Spherical Aberration, Coma, Astigmatism, Curvature of Image,** and **Distortion,** in accordance with the famous "Five Sums of von Seidel." This mathematician has derived a series of *correcting terms*, which must be introduced into the equations derived for ideal image formation. These latter equations are true only when the inclination of the rays with the axis was very small. For rays with greater obliquities, aberrations of the third order largely enter into account, and the elimination of these aberrations necessitates the fulfilment of the five additional sums of von Seidel. Expressing these sums in short by S_1, S_2, S_3, S_4 and S_5, then the condition that each of the five aberrations mentioned above will disappear is that the sums be successively equal to zero. If $S_1 = o$, the axial spherical aberration is eliminated, but the remaining defects may still exist. When $S_2 = o$, coma disappears.* Astigmatism, curvature of image, and distortion are each separately corrected if S_3, S_4 and S_5 respectively equal zero. Then, of a plane object we obtain an image which is sharp, flat, and undistorted; but for *one position* of the object only. If it be required to form correct images of objects at other positions, then additional conditions will have to be satisfied. Even when von Seidel's corrections are applied, the result is not sufficiently accurate and the pracical optician resorts to the less complicated method of direct trigonometrical calculation of the paths of certain rays through each surface in succession. For further information of this method, together with actual examples, *see* Steinheil and Voit's "Handbuch der angewandten Optik," Leipzig, 1891, translated from the German by J. W. French, B.Sc. (published by Blackie & Son, Ltd.).

A summary of von Seidel's work is to be found in "Contributions to Photographic Optics," by O. Lummer, translated by Silvanus P. Thompson, Macmillan, 1900.

Chromatic Aberration.

It will be evident from Equation (30) §26, that the foci of a single lens for the C and F lines of the spectrum will not be coincident, but separated axially. The axial distance between these two foci is called the **Chromatic Aberration** of the lens with respect to these colours. By direct application of Equation (30) the chromatic aberration for a thin lens may be calculated.

Denoting the corresponding focal distances for the C, D and F lines by f_C, f_D and f_F we have

$$\frac{1}{f_D} = (n_D - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

and therefore $\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \left(\frac{1}{n_D - 1}\right)\frac{1}{f_D}$

* This condition $S_2 = o$ corresponds approximately to the fulfilment of the *Sine Condition*. *See* § 49.—Trans.

Also $\frac{1}{f_C} = (n_C - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{n_C - 1}{n_D - 1} \cdot \frac{1}{f_D}$,

$$\frac{1}{f_F} = (n_F - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{n_F - 1}{n_D - 1} \cdot \frac{1}{f_D},$$

i.e., $\frac{1}{f_C} - \frac{1}{f_F} = (n_C - n_F)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{n_C - n_F}{n_D - 1} \cdot \frac{1}{f_D}$.

But since $f_C\, f_F = f_D^2$ *very* nearly, there results

$$\frac{f_F - f_C}{f_F\, f_C} = \frac{f_F - f_C}{f_D^2} = \frac{n_C - n_F}{n_D 1} \cdot \frac{1}{f_D}$$

Hence the chromatic aberration

$$f_F - f_C = \frac{n_C - n_F}{n_D - 1} \cdot f_D = \frac{f_D}{\nu} \quad \ldots \quad (\textit{see } \S 7)$$

Achromatic Lens.

Consider two thin lenses of focal lengths f_D' and f_D'', (the focal length for the D-line is taken as the mean), made of different kinds of glass, whose refractive indices for the various Fraunhofer lines are distinguished by corresponding suffixes and accents. If these lenses are placed in contact, the focal length of the combination for the C-line is

$$\frac{1}{f_C} = \frac{n_C' - 1}{n_D' - 1} \cdot \frac{1}{f_D'} + \frac{n_C'' - 1}{n_D'' - 1} \cdot \frac{1}{f_D''}$$

And similarly for the F-line

$$\frac{1}{f_F} = \frac{n_F' - 1}{n_D' - 1} \cdot \frac{1}{f_D'} + \frac{n_F'' - 1}{n_D'' - 1} \cdot \frac{1}{f_D''}$$

The condition for achromatism is that the foci for the C and F lines be coincident, *i.e.*, $f_F = f_C$. Subtracting the last two equations, the condition for achromatism is :—

$$\frac{n_C' - n_F'}{n_D' - 1} \cdot \frac{1}{f_D'} + \frac{n_C'' - n_F''}{n_D'' - 1} \cdot \frac{1}{f_D''} = 0$$

or

$$\frac{1}{\nu' f'} + \frac{1}{\nu'' f''} = 0$$

or

$$\nu' f' = -\nu'' f'' \quad \ldots \quad \ldots \quad \ldots \quad (A).$$

Since ν' and ν'' have the same sign, f' and f'' must have opposite signs. The two combined will give a lens with a resultant focal length of

$$\frac{1}{f} = \frac{1}{f'} + \frac{1}{f''} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (B).$$

Equations (A) and (B) suffice to determine the condition for achromatism with prescribed focal length.

From (A) $$\frac{f'}{f''} = -\frac{\nu''}{\nu'}$$

also $$\left.\begin{aligned} f' &= \frac{f(\nu' - \nu'')}{\nu'} \\ f'' &= \frac{f(\nu'' - \nu')}{\nu''} \end{aligned}\right\} \quad \cdots \quad \cdots \quad \cdots \quad (C).$$

From Equation (*C*) it will be evident that (if $\nu'' > \nu'$, and f is positive), then

1. f'' is positive.
2. f' is negative.
3. f' must be numerically greater than f''.

As in the case of achromatic prisms, lenses may be corrected for the *C* and *F* lines, and for many practical visual purposes this is sufficient. Owing to the dissimilar run of dispersive constants of crown and flint glasses it is possible to achromatize in this way for only two colours. The utility of the catalogue classifications of the dispersive constants, viz., relative partial dispersions &c. of various glasses in revealing possibilities of achromatisation, will now be more apparent.

Examining once more the conditions for the formation of a positive achromatic lens, the shorter focal length positive element must be constructed from a glass with greater ν-value. Now prior to 1886, only glasses with dispersion increasing proportionally with refractive index were produced and consequently the positive element was made of crown glass, and the negative of flint. Such an achromatic combination is called, "Old" or "Normal" Achromat. The greater variety of glasses introduced by Messrs. Schott in 1886 opened out new possibilities, both in the choice of glass for the simultaneous fulfilment of the conditions for achromatism and flattening of the image (*see* Note, p. 83) and a more complete colour correction. With the older glasses, for example, whilst a lens may be achromatised for the *C* and *F* lines, residual chromatism arises from the dissimilar run of dispersive constants. By employing the newer glasses of Schott & Co. it is possible to obtain a single focus for *three* lines of the spectrum. Consider the curve of chromatic aberration in these two cases. Let the ordinates denote wavelength of light (described by the Fraunhofer lines) and the abscissæ, variations from the focal point. Curve (*a*) illustrates the old type of achromat in which the *C* and *F* lines are brought to one focus. Curve (*b*) shows the chromatic aberration when three spectral lines are brought to a single focus. It is readily seen that the colour union in the second case is much more perfect than in the first. Colour union for two colours leaves a secondary spectrum, whilst colour union for three colours leaves a tertiary spectrum. The older glasses were made with a silica content and it was only by the discovery of the phosphate and borate glasses that such a high colour correction could be obtained. (Fig. 37A.)

A deeper investigation shows that two glasses are most suitable for achromatising when

1. the smaller is the difference between their respective partial dispersions.
2. the greater is the difference between their respective ν-values.

Perfect achromatisation ensues if the ratio of the partial dispersions of one glass to those of the other is equal in value to the ratio of the sum of the curvatures of the surfaces of one lens to that of the other.

It should be observed that as the difference between the ν-values diminishes, the curvatures of the lenses must be increased in order to maintain achromatism.

The scope of the present book does not permit of an exhaustive enquiry into these various aberrations, but the following chapter gives a brief outline of their bearing on the theory of optical instruments.

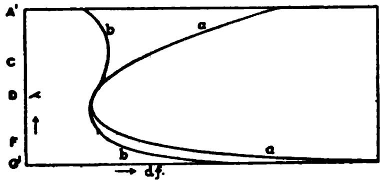

FIG. 37A.

43. Spherical Aberration.

Rays traversing a centred optical system, and more or less remote from the axis, do not, after refraction, intersect exactly in a point.

Fig. 38 shows a plano-convex lens whose focal point P' is the common point of intersection of rays parallel to the axis traversing the paraxial region. Moreover the figure shows two other rays parallel to the axis but more

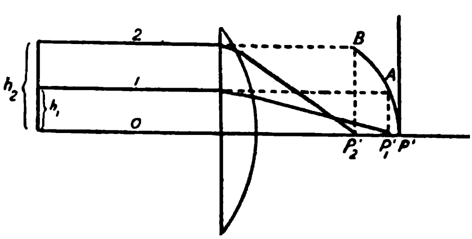

FIG. 38.

remote from it, viz. at "incidence heights" h_1 and h_2. These rays, after refraction, cut the axis in the points P_1' and P_2'. The distances $P'P_1'$ and $P'P_2'$ are called the **spherical aberrations** of the zones of the lens corresponding to the heights h_1 and h_2. If perpendiculars be erected at P_1' and P_2' which cut the rays 1 and 2 produced in A and B, we obtain a graphical representation of the spherical

aberration. The curve through the points $P'AB$ shows clearly the variation of spherical aberration.*

Fig. 39 shows the aberration curves of four forms of lenses each of which has a focal length of 100 mm. The refractive index is 1·5 and the clear aperture amounts to 25 mm. In the graphical representation the scale of both ordinates and abcissae have been doubled.†

* It is proved in works on geometrical optics that the first order spherical aberration of an infinitely thin lens, for incident parallel rays, is

$$d(D) = \frac{h^2}{f} \cdot \left\{ \frac{(2 - 2n^2 + n^3) + \sigma(n + 2n - 2n^3) + \sigma^2 n^3}{2n(n-1)^2(1-\sigma)^2} \right\}$$

where $D = \frac{1}{f} =$ power.

$h =$ incident height.

$n =$ refractive index.

$\sigma =$ ratio of the radii of curvature of the surfaces $= \frac{r_1}{r_2}$.

From this formula the following table of particular cases shows in a concrete manner the actual effects of the various constants involved :—

Lens.	$n = \frac{3}{2}$	$n = 2.$
1st Surface Plane-Plano-Convex	$-\frac{9}{2} \cdot \frac{h^2}{f}$	$-2 \cdot \frac{h^2}{f}$
2nd Surface Plane-Plano-Convex	$-\frac{7}{6} \cdot \frac{h^2}{f}$	$-\frac{1}{2} \cdot \frac{h^2}{f}$
Equi biconvex	$-\frac{5}{3} \cdot \frac{h^2}{f}$	$-\frac{h^2}{f}$
Lens of Least Aberration ... ... (When $n = \frac{3}{2}$ $\sigma = \frac{1}{6}$, when $n = 2$, $\sigma = -\frac{1}{5}$)	$-\frac{15}{14} \cdot \frac{h^2}{f}$	$-\frac{7}{16} \cdot \frac{h^2}{f}$

There is no single lens which gives zero aberration, but it may be shown that least aberration is obtained by a lens of the form II. Fig. 39, in which $\sigma = -\frac{1}{6}$. Such a lens is called a "crossed" lens. It is evident that a plano-convex lens with curved surface facing the incident light is nearly as good as the crossed lens. Further increasing the refractive index decreases the aberration.

Referring to the figure, if the aberration curve lies to the left of the origin, it is said to be under-corrected or uncorrected, if on the right, over-corrected.—Trans.

† *See* W. Zschokke. "Entstehung und Hebung der Bildfehler." Deutsche Mechanikerzeitung. 1910. IX pp. 81–87.

The aberration curves show in each case a continual increase of aberration from the axis to the periphery. The

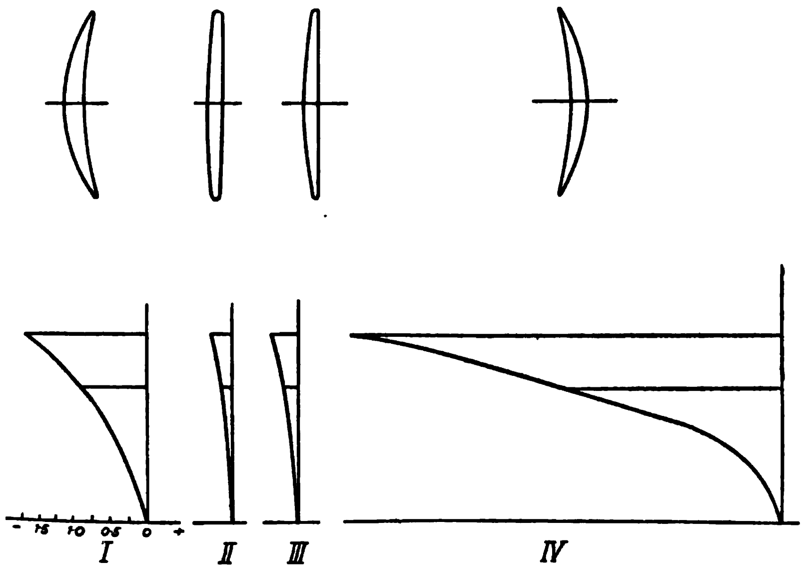

FIG. 39.

aberration is greatest for the meniscus lenses (I and IV Fig. 39), less for the plano-convex (III). A bi-convex lens in which the curvatures of the two surfaces are in the ratio 1 : 6 (II, Fig. 39) is the lens with least aberration.

By means of a suitable combination of several lenses we can arrange that a ray at a definite distance from the axis will pass exactly through the focal point. Then for the corresponding zone, the system is said to be *spherically corrected.* In this case the aberration curve does not recede continuously from the perpendicular to the axis through the focal point, but bends back and cuts this perpendicular in a point which corresponds to the corrected zone (Fig. 40).

If in a particular case the aberration curve degenerates to a straight line lying in the image plane, then the aberration for all zones is completely corrected. It is not possible to obtain image formation free of aberration of this kind by the use of a system of spherical surfaces, at any rate if we

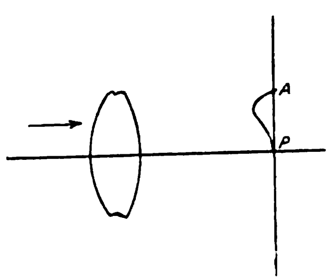

FIG. 40.

wish to obtain a real image point of a real point object (see the chapter on Aplanatism). Yet it is possible, in practice, to eliminate this aberration to a great extent, so that, for simplicity, we shall often make use of the expression "an aberrationless pair of points," although it is not rigidly true.*

44. Astigmatism. First Type of Astigmatic Bundle.

Consider an axial bundle of rays passing from the left to the right and converging on the image point P_m (Fig. 41), as, for example, after refraction through a convex lens. Neglecting spherical aberration, this bundle will be conical, and all cross-sections perpendicular to the axis are circles which become smaller as the image point P_m is approached.

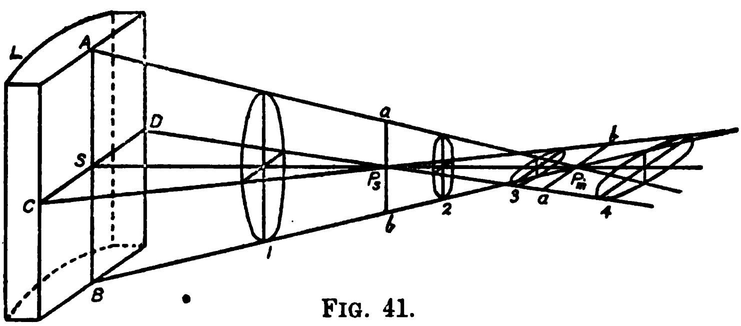

FIG. 41.

A plano-cylindrical convex lens L is placed in the path of the rays. Such a lens consists of a plane surface on one side and a cylindrical surface on the other. The axis of the cylinder and the optical axis SP_m lie in the plane of the paper. This plane will be called the **Meridional Plane,** and the rays travelling in it, the **Meridional Rays.** Evidently the latter rays undergo no refraction in their passage through the lens, since the latter acts as a parallel plate in the meridional section. Further, the plane perpendicular to the plane of the paper and through the optical axis, is called the **Sagittal Section.** The latter cuts the plane surface of the cylindrical lens in the straight line $C\,S\,D$. In the sagittal section, the cylindrical lens acts with a certain power corresponding to the curvature of the cylinder, so that the rays in this plane (the sagittal rays) intersect in a point P_s which lies closer to the lens than P_m.

* It must not be forgotten that, since spherical aberration is influenced by the refractive index, the dispersive action of the glasses introduces further complications—*chromatic differences of spherical aberration.*—Trans.

This property of a bundle of rays to possess different convergences in two planes at right angles is called **Astigmatism.** The point P_m is called the meridional image point, and P_s the sagittal image point, whilst the distance $P_m P_s$ is the **astigmatic difference.**

The astigmatic bundle shown in Fig. 41 is also called the conoid of Sturm.* As will be seen from the figure, planes perpendicular to the axis cut the bundle in ellipses (Sections 1 to 4). At the points P_m and P_s these ellipses degenerate to straight lines—**focal lines**—which are perpendicular to the axis and to each other. The focal line $a\,b$ through the sagittal image point lies in the meridional plane, whilst the focal line through the meridional image point lies in the sagittal plane. The bundle is *completely symmetrical* with regard to the sagittal and meridional planes, and such a bundle is classed as an **Astigmatic Bundle of the first Type.**†

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* J. C. Sturm, Journal de Math. III., Liouville, 1838, was the originator of the theory of astigmatism.—Trans.

† From Fig. 41, in the triangles ASP_m and $a\,P_sP_m$ in the plane of the paper,

$$\frac{P_sP_m}{SP_m} = \frac{aP_s}{AS}$$

$$\frac{P_sP_m}{SP_s} = \frac{bP_m}{cS}.$$

Knowing the constants, powers, etc., of the optical systems, P_sP_m, SP_m; SP_s may be readily calculated, and so aP_s and bP_m are found. By further application of similar triangles, the major and minor axes of any of the elliptical cross sections in any part of the bundle are obtained.

This type of astigmatic bundle is of importance in ophthalmic optics. If we assume this plano-cylindrical (cyl) lens to be in contact with a spherical (sph) lens —indeed, if we work a spherical surface on the plane side of the cylindrical lens, thus obtaining a compound lens with both spherical and cylindrical powers, we have the usual sphero-cylindrical lens (sph-cyl.) of spectacle optics.

Let S be the spherical power in dioptres,
C be the cylindrical power in dioptres,
and θ the angle less than 180° which the axis of the cylinder makes with the horizontal and measured anti-clockwise. Such a lens is described as:

$$S \text{ sph. combined with } C \text{ cyl. ax } \theta^\circ.$$

or contracted to:—

$$S \text{ sph.} \supset C \text{ cyl. ax } \theta^\circ.$$

The power of the sph-cyl. parallel to the axis of the eye is S and the power at right angles to this is $(S + C)$.

Hence the same lens may be expressed in two more different ways as:

$$(S + C) \text{ sph.} \supset -C \text{ cyl. ax } (\theta \pm 90) \text{ where } (\theta \pm 90) \text{ is the corresponding angle less than } 180^\circ$$

or

$$S \text{ cyl. ax } (\theta \pm 90) \supset (S + C) \text{ cyl. ax } \theta^\circ.$$

The powers of the sph-cyl. may thus be expressed in three ways, the process of changing from one expression to another is termed "transposition." The last form is termed a "crossed-cyl."

The results of § 44 may be applied directly to the action of a regularly astigmatic eye, in which most frequently the curvature of the cornea is different in two meridians at right angles to one another.—Trans.

45. Toroidal Surfaces.

The first type of astigmatic bundle, as shown in Fig. 41, was produced by combining the action of a spherical lens with that of a cylindrical lens. It is possible, however, to arrive at a similar effect by means of a single surface which has different curvatures in different directions.

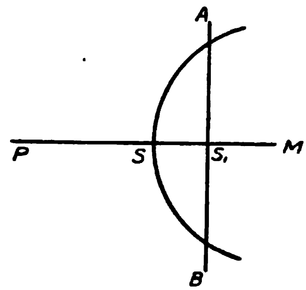

FIG. 42.

Fig. 42 shows a circular arc of centre M and vertex S, which can rotate about the straight line AS_1B as axis (AS_1B not passing through M). The arc then describes a **toroidal** surface which exhibits different refractivities in two perpendicular planes through SM. In the meridional plane (plane of paper) the effective radius is SM, whilst in the perpendicular sagittal plane the effective radius is SS_1, since S_1 in this plane is the centre of curvature. A bundle of rays from P on the axis will be deformed astigmatically by the toroidal surface in the same way as in Fig. 41. If a toroidal (or toric) surface be combined with a second toric or with a spherical surface, then double toric or toric spherical lenses respectively are obtained.*

46. Second Type of Astigmatic Bundle.†

In Fig. 43, the ray PA from the point P is incident on a refracting spherical surface whose centre is M. The refracted ray cuts PM produced in P_s', where P_s' is the sagittal image point of an infinitely thin bundle about PA. This will be clear if Fig. 43 be assumed to undergo a small rotation round the axis PMP_s', so that the point A moves from a position just below to one just above the plane of the paper. The point P_s' is unchanged by this rotation and the rays of the infinitely thin bundle PA, which lie symmetrically above and below

* *Toric Lenses.*—In § 148, reference is made to the periscopic lenses of Wollaston (Feb. 9, 1804). These were meniscus lenses with the concave surface inwards, the general aim of which was to yield equally sharp images of objects seen through both the central and peripheral portions of the lens. Certain modifications have superseded the Wollaston forms, and thin spherical lenses called deep menisci are now manufactured with commercial success.

By a toric lens in ophthalmic work is understood a lens one surface of which is a deep spherical, whilst the other is a toroidal surface. The spherical surface is termed the base curve and the toric surface is described in prescriptions as a cross-cyl. The resulting lens resembles in outward appearance a deep meniscus, and frequently this latter term is wrongly applied to it. A deep meniscus is a spherical lens only, whilst a toric contains in addition a cylindrical or toroidal element.—Trans.

† Radial astigmatism or astigmatism due to obliquity.

the plane of the paper, intersect in P_s'. The plane of the paper represents the meridional plane. A ray PA_1 *very* close

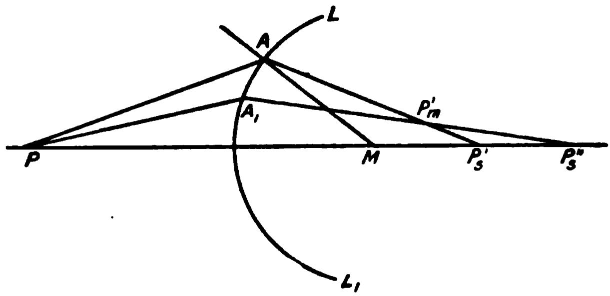

FIG. 43.

to PA, cuts the axis after refraction in P_s'' and the first refracted ray in P_m', which point represents the meridional image point. It will be seen that on account of the above rotation, the sagittal rays near P_m' traverse a very short focal line perpendicular to the plane of the paper, whilst the meridional rays form the focal line $P_s' P_s''$ at the position of the sagittal image point.

This latter line is *not* however, *perpendicular to the direction of the refracted pencil*, in which particular the second type of astigmatic bundle differs in principle from the first. It can be seen from Fig. 43 that in the sagittal plane complete symmetry prevails as in the first type of bundle. This is not the case in the meridional plane. A third ray from P in the plane of the paper and very close to PA, does not pass through P_m', and in place of the meridional image point, we obtain a very small portion of a caustic: that is, the locus of contiguous luminous points which are the points of intersection of neighbouring rays taken in pairs. The union of rays in the sagittal section is of a higher order than that in the meridional section.

What applies to a single refracting spherical surface applies also to a system of such surfaces. A bundle of rays traversing a principal plane of an optical system and lying outside the paraxial region is thus an **astigmatic bundle of the second type.**

For a more exact statement of astigmatism we are indebted to Gullstrand.*

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* *See* the works of this author:
Skandinav. Archiv für Physiologie, 2, 1891.
Nova acta regiae soc. Ups. scientc. Ser. 3. Vol. XX.
Archiv für Optik. Pt. I.
In these works a third form of astigmatism is dealt with, which must, however, be omitted here.
See also Southall's Geometrical Optics, Chapter XI.—Trans.

47. Graphical Representation of Astigmatism.

Astigmatism is defined as a property of *infinitely thin* bundles. Theoretically such a bundle may be obtained by employing an infinitely narrow diaphragm. The images of this diaphragm formed to the right and left by the intervening parts of the system, yield the infinitely small entrance pupil and exit pupil whose centres are R and R' respectively in Fig. 44. The point R is frequently called the Aperture or Stop Position.

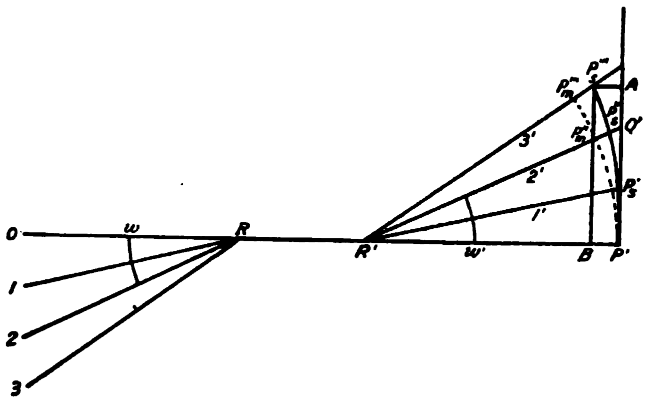

FIG. 44.

Through R and R' a number of rays are drawn which are the axes of the bundles traversing the system. Such rays are usually called **Principal Rays.*** For simplicity we will assume that R and R' are a pair of aberrationless points. On the principal rays 1′, 2′, 3′, in the image space, which correspond to the object space principal rays 1, 2, 3, the meridional image points P'_m, P''_m and P'''_m are indicated, as are also the sagittal image points P'_s, P''_s and P'''_s. Through the former is traced a dotted line and through the latter a full line curve. Both curves pass through the image point P' formed by the rays in the paraxial region. For distant objects, this point is the focal point. Further, no astigmatism exists at this point. The distance along a principal ray between the astigmatic image points P'''_m, P'''_s; P''_m, P''_s &c., is called the **Astigmatic Difference.** If this difference is zero, then the system is **astigmatically corrected** for the particular obliquity of principal rays under consideration. If the coincident meridional and sagittal image points fall on an image or focal plane through P' perpendicular to the axis,

* Compare S. P. Thompson: Photographic Optics by Lummer. Macmillan. 1900. Note, p. 31.—Trans.

the system is **anastigmatically flattened** for the particular obliquity of rays considered. In Fig. 44 the horizontal distance of an image from the focal plane (*e.g.*, $P_{,}'''A$) is represented by z; the vertical distance from the axis (*e.g.*, $P_{,}'''B$) by y.

The curves, shown in Fig. 45, represent the astigmatic conditions for four forms of lenses. The table gives the construction data of the lenses, radii of curves, position of stop, &c. The focal length of each lens is 100 mm.

ASTIGMATIC ERRORS OF SIMPLE LENSES.

——	I.	II.	III.	IV.
1 Radius	+ 58·0	− 30·0	− 12·0	− 4·78
2. Radius	−359·38	− 19·06	− 12·0	− 4·81
Thickness	1·5	1·5	6·4	1·2
Refractive Index ...	1·5	1·5	1·6	1·51
Angle of Inclination of ray in object space.	15°	15°	15°	30°
Position of stop R ...	− 5·45	− 6·35	− 5·58	− 3·84
y meridional	+ 23·66	+ 26·09	+ 26·40	+56·67
y sagittal	+ 25·21	+ 26·09	+ 26·32	+56·67
z meridional	− 11·68	− 2·17	+ 0·69	− 0·24
z sagittal	− 5·51	− 2·17	+ 0·35	− 0·24

It will be noticed that the values of z are reckoned negative to the left and positive to the right of P' in Fig. 44 (P' is designated 0 in Fig. 45). Also the diaphragm positions are reckoned negative to the left and positive to the right from the vertex of the lens.

Since infinitely narrow diaphragms are never actually used, the effect of the position of the diaphragm cannot be stated with any accuracy: in the case of a wide angle system there is no guarantee that the infinitely thin bundle of rays passing through the assumed diaphragm position actually passes accurately through the meridional image point, and hence that the calculated astigmatic condition represents the actual state of correction. With regard to the astigmatic conditions, the position of the stop (position of entrance pupil) is influenced in wide angle systems by certain practical con-

siderations. The author has evolved a method by which this diaphragm position can be estimated to a first approximation, on the assumption that the principal rays in the image space pass through a certain mean position.*

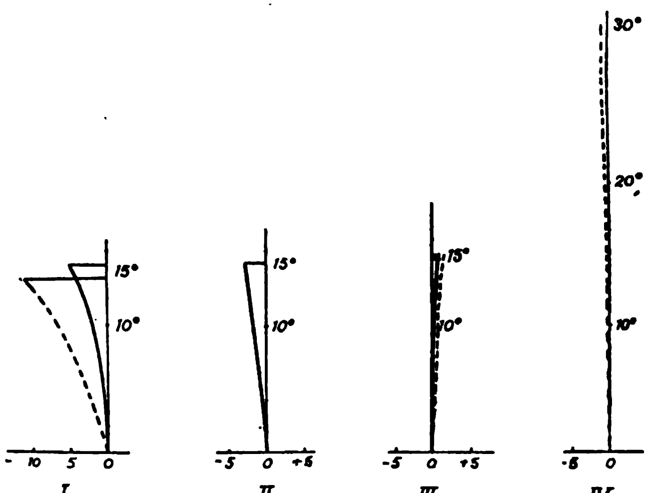

FIG. 45.

If we choose in turn various diaphragm positions for one and the same system, then each time, on calculation, we obtain a different astigmatic condition and a different image curvature. In Fig. 45, Diagram I. shows the correction condition for a bi-convex lens of almost plano-convex form, whilst the Diagrams II., III., IV. represent meniscus lenses with their concave surfaces facing the incident light (*see* the construction data in the table above).

A comparison of Figs. 39 and 45 shows plainly that the form of lens which exhibits the greatest astigmatic error possesses the smallest spherical aberration and *vice versa.*

By a suitable choice of various kinds of glass and curvatures of surfaces, a large number of optical systems have recently been calculated (especially photographic objectives) in which both spherical and astigmatic errors have been corrected.†

* *See* "Vorlesungen über photographische Optik." Leipzig. 1905, p. 136, and Hinrichs "Beitrag zur Theorie der natürlichen Blende optischen Instrumente." Diss. Rostock. 1910.

† If it be assumed that a system is astigmatically corrected, so that the image of a plane object is formed on a curved surface, then the condition for flattening is stated in § 36 (Petzval Law). The Seidel sums S_1, S_2, and S_3 must respectively be

48. Orthoscopy of a System.

Consider the aberrationless points R and R' in Fig. 44 as centres of perspective. If the image be not anastigmatically flattened, the points such as Q', in which the principal rays $1', 2', 3'$ cut the image plane through P' perpendicular to the axis, must be considered as image points. Obviously they will be the centres of the blur circles on this plane.

Let $$P'Q' = y' \qquad Q'\hat{R'}P' = \omega' \text{ and } R'P' = p'$$

then
$$\tan\omega' = \frac{y'}{p'}.$$

Similarly in the object space:
$$\tan\omega = \frac{y}{p}$$

where p is the axial distance of the foot of the object measured from R.

Hence
$$\frac{\tan\omega'}{\tan\omega} = \frac{y'}{y} \cdot \frac{p}{p'} \qquad \ldots \qquad \ldots \qquad (1)$$

If the centres of the pupils be assumed to be aberrationless, then the ratio $\frac{p}{p'}$ is constant for a particular pair of conjugate planes for all inclinations of the rays, and in this case the constancy of the ratio $\frac{y'}{y}$, which determines the similarity* of object and image, depends on the **tangent relation** $\tan\omega' : \tan\omega$.

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equal to zero before this law, which corresponds to the fourth sum of Seidel, can be applied. It is easily seen from § 36 that the condition for a flat image to be produced by a combination of two lenses is—

$$n_1 f_1 = - n_2 f_2.$$

Hence

1. f_1 and f_2 must have opposite signs;
2. the lens with smaller f must have the higher n.

Now the condition for achromatism with the older glasses (prior to 1886) required f for the flint to be greater than f for the crown, or the lens with greater f must have the higher n. (Footnote following § 28.) This is directly opposed to the flattening condition. Hence the older achromatic objectives of necessity could not produce a flat image. The newer glasses of Messrs. Schott & Co. made possible the simultaneous fulfilment of achromatism and flattening of the image (*see* §§ 142 and 143). Anastigmatic objectives were thus rendered possible, and with them a wide field was opened for photographic objectives. The older achromatic objectives were termed "normal" by Rudolph or "old" by Lummer, whilst the new types were called respectively "anomalous" or "new."—Trans. *See* Zschokke, "Anschaliche Darstellung der Entstehung und Hebung der Bildfehler." Deutsche Mechanikerzeitung, 1910, Vol. x., p. 81-87.

* An image which is orthoscopic is one which is free from *distortion*, that is to say, straight lines or figures in an object plane will be imaged as geometrically similar straight lines and figures in the image plane. The measure of similarity is expressed

Actually, however, in an optical system, especially one of wide aperture, the aberration at the pupils for finite inclinations ω and ω' is never quite eliminated. Assuming that for rays in the paraxial region, p and p' have the values p_0 and p_0' and that the aberrations for the inclinations ω and ω' are respectively δ and δ', then :

$$p = p_0 + \delta$$
$$p' = p_0' + \delta'$$

and equation (1) becomes :

$$\frac{\tan \omega'}{\tan \omega} = \beta \cdot \frac{p_0 + \delta}{p_0' + \delta'}$$

which is the condition for **Orthoscopy**, and may also be expressed :

$$\frac{\frac{\tan \omega'}{\tan \omega} \cdot \frac{p_0' + \delta'}{p_0 + \delta}}{\beta} = 1$$

The amount by which this expression differs from unity is taken as a measure of the distortion V, that is,

$$Distortion = \mathrm{V} = \frac{\frac{\tan \omega'}{\tan \omega} \cdot \frac{p'_0 + \delta'}{p_0 + \delta}}{\beta} - 1 \quad \ldots \quad (1a)$$

For $\beta = 0$, that is for infinitely distant objects, $(p_0 + \delta)$ becomes infinitely great, but the expression $(p_0 + \delta)\beta$ in the limit becomes equal to f. This will be seen from the

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by the constancy of the ratio $\frac{y'}{y}$ for different inclinations ω of the incident principal rays. It may so happen in an optical system that with an increase of ω, the ratio $\frac{\tan \omega'}{\tan \omega}$ may either increase or decrease correspondingly. Now since from equation (1), $\frac{y'}{y}$ varies directly as $\frac{\tan \omega'}{\tan \omega}$, the magnification $\frac{y'}{y}$ may either increase or decrease with an increase of the inclination ω Consider the case when the ratio $\frac{\tan \omega'}{\tan \omega}$ increases with increase of the angle ω. The magnification $\frac{y'}{y}$ will consequently increase towards the margins of the field and we obtain the distortion known as *Cushion Shaped or Positive* distortion. On the other hand if $\frac{\tan \omega'}{\tan \omega}$ decrease with increase of ω, the magnification decreases towards the margins of the field and we have the *Barrel Shaped or Negative* distortion. For a further treatment of orthoscopy *see* Geometrical Optics by Southall.—Trans

following considerations. Let x_0 be the distance of the object from the first principal focus and ξ_0 the distance of this focal point from the centre of the Entrance Pupil, then

$$p_0 + \delta = x_0 + \xi_0$$

where ξ_0 has always a finite value. Now

$$\beta = \frac{f}{x_0}$$

and consequently the last equation may be written in the form

$$(p_0 + \delta)\ \beta = f + \xi_0 \beta$$

Substituting $\beta = o$, we obtain as stated above

$$(p_0 + \delta)\ \beta = f$$

The condition of orthoscopy for distant objects thus takes the form

$$V_{(\beta=0)} = \frac{tan\ \omega'}{tan\ \omega} \cdot \frac{p_0' + \delta'}{f} - 1$$

This equation is taken as the basis for the treatment of the orthoscopic condition of the photographic objective.*

49. The Sine-Condition.

If an optical system form an aberrationless image point P' of a point object lying on the axis, it does not follow that an object at P perpendicular to the axis will form an aberrationless image at P'. The **Sine-Condition**, however, due to Abbe, states the condition for the formation of an image free from aberration—at any rate for small objects perpendicular to the axis. Fig. 46 represents a centred optical system whose vertices are S_1 and S_2 and a pair of aberrationless points P and P'. A very small object y perpendicular to the axis at P, forms an image at P' of size y', by means of a system having a *finite aperture*.

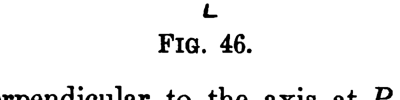


FIG. 46.

Consider further any ray from P whose inclination angle is $S_1\hat{P}A = u$ which passes through the point P' at the inclination $S_2\hat{P'}A' = u'$, then the sine-condition may be expressed :

$$\frac{sin\ u'}{sin\ u} = \frac{n}{n'} \cdot \frac{1}{\beta} \qquad \dots \qquad \dots \qquad (2)$$

* See Zeitschr. *für* Instrumentenkunde, 1907. p. 77 &c., the work of Wandersleb.

where n and n' are the refractive indices of the object and image spaces and $\beta = \frac{y'}{y}$ is the lateral magnification at the points P and P'. Since the right hand side of this expression is constant for the particular pairs of points considered, the sine-condition may be expressed in the form :

For all rays filling the aperture, the ratio of the sines of the inclination angles of each pair of corresponding incident and refracted rays shall be constant.

Points free from aberration are called by Abbe **Aplanatic points** only if the sine-condition is also fulfilled.*

50. The Case of Distant Objects.

If the point P be very distant, and a ray from P cuts the first focal plane in l, then $PF = x$ and $lF = h$ and since *sin u* may be written for *tan u*, then $\sin u = \frac{h}{x}$ and equation (2) §49 then becomes

$$\sin u' = \frac{n}{n'} \cdot \frac{1}{\beta} \cdot \frac{h}{x} \quad \ldots \quad \ldots \quad (3)$$

Since $\frac{n}{n'} = \frac{f}{f'}$ and $\beta = \frac{f}{x}$ where f and f' are the first and second focal lengths of the system, the last equation becomes

$$\frac{h}{\sin u'} = f' \quad \ldots \quad \ldots \quad (4)$$

and is the condition that the sine-condition be fulfilled for an infinitely distant object. In this case the ray PA is parallel to the axis and distant h from it. The incident and emerging rays intersect if produced, in N (Fig. 46), and the perpendicular on the axis from N is $NE = h$. From the triangle $P'NE$ we have

$$\sin u' = \frac{h}{NP'}$$

Equating the last two equations

$$NP' = f' \quad \ldots \quad \ldots \quad (5)$$

Hence the rule :

For an infinitely distant object the sine-condition is fulfilled if the points of intersection of all incident rays parallel to the axis with their corresponding emerging rays lie on a circle of radius f' and centre F', the second focal point.

* For the proof of the sine-condition, *see* the Author's—Lehrbuch der geometrischen Optik, Leipzig, 1902, p. 172; and Vorlesungen über photographische Optik, Leipzig, 1905, p 109.
See also "Photographic Optics" by Lummer; Appendix II.—Trans.

51. Theorem on Sine-Condition.

The last rule is a special case of the following theorem: If the incident and emerging rays of inclinations u and u' be produced to intersect in N (Fig. 46) and if this construction be repeated for rays of all possible inclinations, then the locus of N is a circle. This Appollonian circle has the characteristic property that it divides the line PP' internally and externally in the same ratio. This ratio is equal to $\frac{n}{n'\beta}$, where β is the lateral magnification at the points P and P'.

In Fig. 47, K and K' are the poles of the circle above referred to, M is the centre, P and P' are the conjugate points, whilst the ray PNP' makes the divergence angles u and u' with the axis.

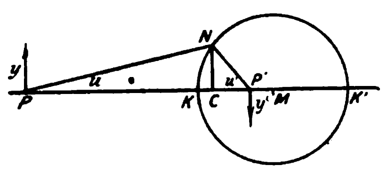


FIG. 47.

From geometry it may be shown that

$$\frac{PK}{P'K} = \frac{PN}{P'N}$$

Dropping the perpendicular NC,

$$sin\ u = \frac{NC}{PN} \qquad sin\ u' = \frac{NC}{P'N}$$

By division

$$\frac{sin\ u'}{sin\ u} = \frac{PN}{P'N} = \frac{PK}{P'K} = constant$$

thus proving the above law.

The constant in the last equation is $\frac{n}{n'\beta}$ which we may write as $\frac{1}{w}$.

Then
$$\frac{1}{w} = \frac{n}{n'}\cdot\frac{1}{\beta} \qquad \dots \qquad \dots \qquad (6)$$

If $PP' = e$, then it is easily seen that

$$KP = \frac{e}{1+w} \qquad KP' = \frac{ew}{1+w}$$

and for the radius

$$KM = NM = \frac{ew}{1-w^2}$$

The distance of P' from the centre M is

$$MP' = \frac{ew^2}{1-w^2}.$$

52. Special Cases.

If $w=1$, then $\beta=\frac{n}{n'}$ and the circle degenerates to a straight line perpendicular to the axis at a distance $\frac{e}{2}$ from P.

If w is much greater than unity we may write the above expressions

$$KP = \frac{e}{w}; \ Radius\ KM = -\frac{e}{w}; \ MP' = -e.$$

The centre of the circle then lies at the object point P and its radius has the numerical value $\frac{e}{w}$. The sign of KM signifies that the point M lies to the left of K. For a microscope objective of a hundred magnification with object and image points at a distance 200 mm. apart, this radius is 2 mm.; therefore the sine-condition fixes a limit (of 4 mm.) for the aperture of the objective in question.

Translators' Appendix on Sign Convention.

British optics at the present time suffers considerably from a lack of uniformity in treatment. The subject as treated from the standpoints of the photographer, the optician, the university student, &c., is hardly recognizable as being one and the same subject. Such a confusion arises partly from the method of treatment and partly from the terminology adopted in these various branches of opto-technics. Hence the need is frequently expressed, especially in the case of the more elementary treatises, for some consistent terminology and a convention which will indicate the direction of measurement in such a way as to permit of ready application both in practice and also to many of the standard books on the subject. Dr. A Gleichen, on account of a similar confusion arising in Germany, has endeavoured to introduce a system of applied optics aiming at a compromise between the requirements of the various branches of technical optics.

A consistent method of indicating the direction of measurement in a system of optics is the first essential consideration, and many of the existing British conventions do not find approbation from the more practical opticians. For example, it is frequently found stated in books that a "positive" converging lens has a "negative" focal length, that the direction in which the light travels is the negative direction, &c. Sometimes in other books, the reverse is the case, that the direction of the light is the positive direction, &c. Such a variety of impositions tends to confuse the beginner.

It will be observed that the underlying aim of the convention, adopted here by Dr. Gleichen in the form of *Normal Figures*, reduces most distances which enter into more common practice, to positive distances. A positive lens has a positive focal length; image inversion and the formation of real images, which is so common in many lens actions is thus designated positive. Such a convention

commends itself to the practical man on account of its freedom from mathematical abstractions and also on account of the visualization entailed in the applications of these normal figures. Even with this convention uncertainty arises, as will be seen below, in connection with the direction of measurement of angular ratios. The following summary is intended to indicate the general underlying plan linking the various normal figures.

1. Light is always to be considered as passing *from left to right*, this being the positive direction for the light to travel.

2. The *positive* direction for the measurement of the distance of an axial object point from its corresponding point of reference in the object space, is towards the *left*, and is consequently opposed to the direction of the incident light. Image distances on the other hand are measured *positive* to the *right* of the point of reference, *i.e.*, in the same direction as the incident light. (*Cf.* Fig. 11.)

3. An object is *erect* and *positive*, if it lies above the axis. An image is *inverted* and *positive* if it lies below the axis. (*Cf.* Fig 11.)

4. *Angles in both the object and image spaces are positive if measured in the clockwise direction.*

A certain amount of uncertainty arises in the case of angular magnitudes in the text, owing to the fact that the signs of the angles in Fig. 12 are not compatible with those of Fig. 11. Thus in Fig. 12 the angles u and u' are measured in opposite directions. This convention is not consistent with Fig. 11 or Fig. 13, which require that both u and u' be measured positive in the same direction, viz., the clockwise direction. Thus in Fig. 13

$$u=\frac{y}{a} \text{ and } u'=\frac{y'}{a'}$$

the angles u and u' must be positive since y, a, y', and a', are all positive. This explains the discrepancy between equation (17) :— $\gamma=\frac{u'}{u}=-\frac{n}{n'}$ or $n'u'=-nu$, and the equation immediately following, *i.e.* $nu=n'u'$. To preserve the consistency in the convention then, we have the above *revised* convention for angular quantities.

Equation (16*a*) § 19, now becomes :

$$\frac{u'}{u}=\gamma=-\frac{1}{\beta}\cdot\frac{n}{n'}$$

and when $\beta=-1$, $nu=n'u'$.

Further for the nodal points, where $u'=u$

$$\gamma=1=-\frac{1}{\beta}\cdot\frac{n}{n'} \text{ and since } \beta=\frac{f}{x}=\frac{x'}{f'} \text{ and } \frac{n}{n'}=\frac{f}{f'}$$

$$x=-f' \text{ and } x'=-f$$

Trans.

PART II.

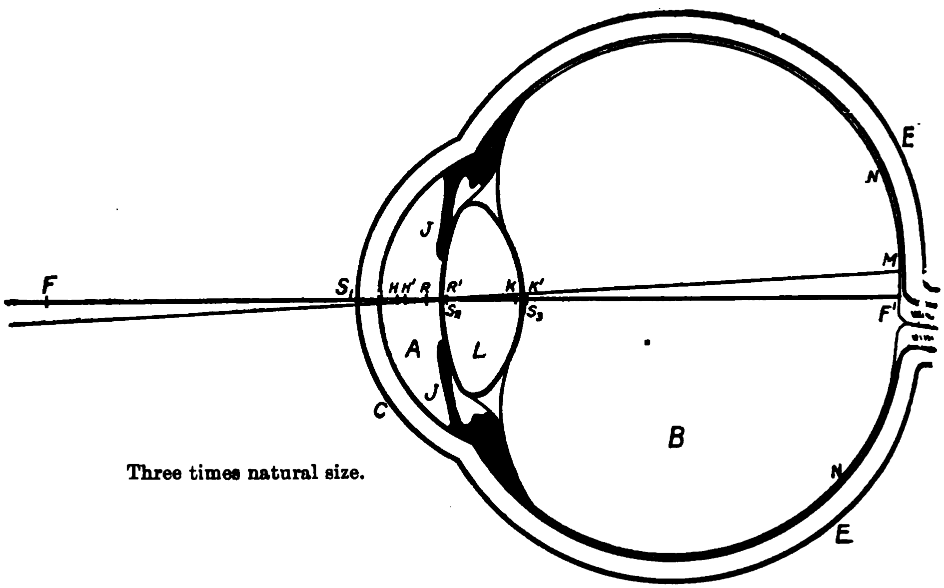
OPTICAL INSTRUMENTS.

CHAPTER VI.

The Human Eye.

53. Description of the Eye.

Fig. 48 represents a horizontal section of the right eye of a human being, with first focal point F and second focal point F'. FF' is the optic axis. In the direction of the axis from the left we have first the **Cornea,** C, which merges behind into the opaque **Sclerotic,** E, surrounding the remainder of the eye. The latter is penetrated behind,



Three times natural size.

FIG. 48.

somewhat to the nasal side, by the **Optic Nerve,** G. The thickness of the cornea is about 1 mm.; the radius of its inner (posterior) surface is smaller than that of the outer (anterior), so that the cornea, if it were surrounded by air.

would act as a diverging lens. Behind the cornea is the **Anterior Chamber,** A, filled with a watery fluid, bounded behind by the **Iris,** J; this has a central circular aperture, the **Pupil,** immediately behind which is the anterior surface of the eye lens—called also the **Crystalline Lens**—composed of a series of fibres arranged in layers. This lens is bi-convex, and, when in a condition of rest, the anterior surface is considerably less curved than the posterior. Behind the crystalline lens is the large **posterior chamber,** B, filled with a transparent gelatinous medium, called the **Vitreous Humour.** This large chamber is surrounded, on the outside, by the outer layer, the sclerotic, within which is the **Choroid,** of a dark colour, and merging in front into the iris. Over the inside layer of the choroid spreads the **Retina,** which is of complicated structure, and may be conceived as a continuation of the optic nerve. The retina is the screen, sensitive to light, on which the images of external objects are formed, corresponding to the sensitised plate in the camera. At the point where the optic nerve enters the eye, the retina is not sensitive to light impressions; this spot is the **Blind Spot** (macula coeca). The part most sensitive to light is the **Yellow Spot** (macula lutea); it does not coincide with point F', where the optic axis cuts the retina, but lies towards the temporal side. The central depression or pit, M, of the yellow spot, is called the **Fovea Centralis,** and is that point on which images are formed of objects that are under observation (fixed) by the eye.

On the axis are the cardinal points—focal points F and F', principal points H and H', nodal points K and K', and the centres R and R' of the entrance and exit pupils. The line drawn from M through R', which is continued through R and out into space, is called the **line of vision** or **visual axis.** It connects the point under observation (point of fixation) to the position of most distinct vision on the retina.

We have already given, § 33, the data of the schematic eye of **Helmholtz.** Data of this kind for schematic eyes have been drawn up by many investigators, particularly by **Gullstrand.***

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* *See* Helmholtz, "Handbuch der physiologische Optik, 3rd. Ed., Hamburg and Leipzig, 1909. Vol. I, p. 300.

The schematic eye data of Gullstrand given in this work are tabulated below. With regard to the term "equivalent nucleus lens," it should be remembered that the crystalline lens is not homogeneous but is built up of a great number of layers, the refractive indices of which increase continuously towards the "*nucleus*," or point

Since in the treatment of schematic eyes we are concerned only with approximate mean values, we append a Table giving Helmholtz's values (later form). The heading "far" refers to the eye focussed on infinity; the heading "near" to the eye accommodated to a point 140·33 mm. from the first focal point, *i.e.*, 152·5 mm. from the vertex of the cornea.

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of greatest optical density. The refractive index at this point has been derived. Helmholtz in his work using the value 1·406 as the index of the equivalent nucleus lens by which he replaces the complex crystalline lens.

Schematic Eye at Rest—Gullstrand.

The eye is at rest, *i.e.* focussed on infinity; these values thus correspond to those under the heading "far" in Helmholtz's table p. 93.

Refractive Index of Cornea	1·376	
„ „ „ Aqueous and Vitreous Humours	1·336	
„ „ „ Crystalline Lens	1·386	
„ „ „ Equivalent Nucleus Lens	1·406	
Position of anterior surface of cornea	0·0	mm.
„ „ posterior „ „ „	0·5	„
„ „ anterior „ „ crystalline lens	3·6	„
„ „ „ „ „ equivalent nucleus lens	4·146	„
„ „ postrior „ „ „ „ „	6·565	„
„ „ „ „ „ crystalline leos	7·2	„
Radius of anterior surface of cornea	7·7	„
„ „ posterior „ „ „	6·8	„
„ „ anterior „ „ crystalline lens	10·0	„
„ „ „ „ „ equivalent nucleus lens	7·911	„
„ „ posterior „ „ „ „ „	− 5·76	„
„ „ „ „ „ crystalline lens	− 6·0	„
Power of anterior surface of cornea	48·83	Dp.
„ „ posterior „ „ „	− 5·88	„
„ „ anterior „ „ crystalline lens	5·0	„
„ „ equivalent nucleus lens	5·985	„
„ „ posterior surface of crystalline lens	8·33	„

Cornea—

Power	43·05	„
Position of first principal point	− 0·0496	mm.
„ „ second „ „	− 0·0506	„
First focal distance	−23·227	„
Second focal distance	31·031	„

Crystalline Lens System—

Power	19·11	Dp.
Position of first principal point	5·678	mm.
„ „ second „ „	5·808	„
Focal distance	69·908	„

Complete System of Eye—

Power	58·64	Dp.
Position of first principal point	1·348	mm.
„ „ second „ „	1·602	„
„ „ first focal point	−15·707	„
„ „ second focal point...	24·387	„
Anterior focal distance...	−17·055	„
Posterior „ „	22·785	„
Position of Fovea	24·0	„
Axial Hypermetropia	1·0	Dp.

Trans.

The first seven values of the table are given; the remainder calculated. The positions are referred to the vertex of the cornea.

SCHEMATIC EYE—HELMHOLTZ.
DISTANCES IN MILLIMETRES.

——	Far.	Near.
Refractive index of aqueous and vitreous humour	1·3365	1·3365
Refractive index of crystalline lens ...	1·4371	1·4371
Radius of curvature of cornea	7·829	7·829
Radius of anterior surface of crystalline lens	10	6
Radius of posterior surface of crystalline lens	− 6	− 5·5
Position of anterior surface of crystalline lens S_2	3·6	3·2
Position of posterior surface of crystalline lens S_3	7·2	7·2
First focal distance of eye	15·5	14·0
Second „ „ „	20·71	18·69
Position of first focal point	− 13·74	− 12·13
„ second „	22·82	20·95
„ first principal point... ...	1·75	1·86
„ second „ „	2·10	2·26
„ first nodal point	6·97	6·57
„ second „ „	7·32	6·97
„ centre of entrance pupil ...	3·046	2·67
„ „ exit pupil	3·705	3·298
Magnification in entrance pupil and exit pupil	0·923	0·941

54. Emmetropia. Size of retinal Image of emmetropic eye.

By an **emmetropic** eye is meant one, which, when in a condition of rest, focusses infinitely distant objects sharply on the retina. *Thus the vergence in the object space of an emmetropic eye at rest is zero.* If the eye, by shortening the focal length of the crystalline lens,—*i.e.*, by **Accommodation**—focus up objects lying nearer, then the vergence becomes positive.

The vergence of the eye is known as the **Refraction** of the eye. The term Refraction by itself is used to denote the vergence referred to the first principal point of the eye, whilst the vergence referred to the first focal point is called the **Focal point Refraction**, or, for reasons to be dealt with later, the **Spectacle Refraction.**

We will represent the latter by the letter L, so that L is identical with the magnitude X in § 34. If we represent the power of the eye generally by D, the formulæ in §§ 32–40 above become :

$$A + A' = D$$

$$B + B' = D_K = \frac{D}{n'} \quad (n' = \text{Refractive index of vitreous humour}).$$

$$Q + Q' = D\rho\rho'$$

$$L\,L' = D^2$$

according as we take as our origin of measurement the principal points, the nodal points, the pupils or the focal points.

For an emmetropic eye at rest, that is unaccommodated, the magnitudes A, B, Q, L, representing the refraction, are equal to zero. Hence from the above equations we can state the condition for emmetropia for an eye at rest in four different ways :

$$A' = D, \qquad B' = \frac{D}{n'}, \qquad Q' = D\rho\rho', \qquad L' = \infty$$

Each of these equations states the condition that the system of the eye must be so arranged that the retina is conjugate to an infinitely distant plane perpendicular to the axis. From each of these equations we can calculate the distance from that point where the optic axis cuts the retina to the origin of measurement in the image space. The point where the optic axis intersects the retina we will call briefly the retinal point.*

Ex. 53.

What does the condition $A' = D$ express ?

$$\text{Since } A' = \frac{n'}{a'} \text{ and } D = \frac{n'}{f'} \text{ then } a' = f'$$

that is, the distance of the second principal point from the fovea is equal to the second focal distance.

Ex. 54.

What does the condition $B' = \frac{D}{n'}$ express ?

$$\text{Since } B' = \frac{1}{n'b'} \text{ and } D = \frac{1}{f} \text{ then } \frac{1}{n'b'} = \frac{1}{n'f}$$

that is $b' = f$, or the distance of the second nodal point from the fovea is equal to the first focal distance.

* The *fovea centralis*, the part of the retina most sensitive to light impressions lies a little towards the temporal side.

Of all the expressions for finding the lateral magnification β, the simplest from which to find the size y', of the retinal image of an object y is

$$y' = y\frac{f}{x}$$

where f is the first focal distance of the eye, and $\frac{1}{x} = L$, the focal point refraction of the eye.

Infinitely distant objects subtending the visual angle ω, produce a retinal image of the size

$$y' = f\,\omega$$

Ex. 55.

What is the size of the retinal image of a child whose height is 1 m., situated 15 m. from the eye?

$y = 1$ m.

$x = 15$ m. $\quad \therefore y' = \frac{0\cdot015}{15} = 0\cdot001$ m. $= 1$ mm.

$f = 0\cdot015$ m.

Ex. 56.

A distant balloon subtends at the eye the angle $u = 1'$. What is the size of the retinal image?

$\omega = 0\cdot00029$ radian

$\therefore y' = 0\cdot015 \times 0\cdot00029$ m. $= 0\cdot0044$ mm.

To eyes of normal vision, this object can still be distinguished from a point.

55. Emmetropia and Ametropia.

That point on the optical axis which is conjugate to the retinal point when the eye is at rest (unaccommodated) is called the **Far Point.** Thus the unaccommodated eye is focussed sharply on the far point. If the far point lie at infinity, the refraction of the eye is equal to zero, and the eye is **emmetropic.** In all other cases the eye is said to be **ametropic**, the corresponding refraction condition of the eye being called **Ametropia.** If the far point lie at some finite distance in *front* of the eye, so that it actually exists in space, we have **Myopia,** or short-sightedness. In this case *divergent* rays from a point in front of the eye are brought to a focus on the retina. If, on the other hand, the far point lie *behind* the eye, so that it is virtual, we have **Hypermetropia,** or long-sightedness. In this case, those rays are brought to a focus on the retina that are *converging* on a point behind the eye.

Figs. 49, 50 and 51 represent diagrammatically an emmetropic, a myopic and a hypermetropic eye. P is the far point, N the retinal point, F and F' the first and second focal points, and H the first principal point of the eye. In Fig. 50 put

$$PH=a$$

Then $A=\frac{1}{a}$ is the refraction of the eye, referred to the first principal point.

Let $$PF=l \ldots \quad \ldots \quad \ldots \quad (1)$$

then $$L=\frac{1}{l} \ldots \quad \ldots \quad \ldots \quad (2)$$

is the refraction of the eye referred to the first focal point or the **Spectacle Refraction.***

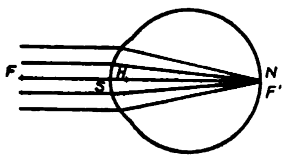

FIG. 49.

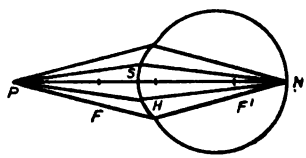

FIG. 50.

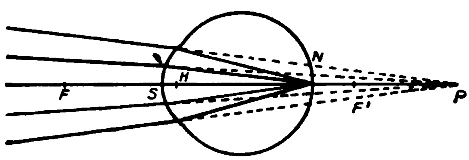

FIG. 51.

Since FH is the first focal distance of the eye, we have

$$a-l=f$$

or $$\frac{1}{A}-\frac{1}{L}=f \quad \ldots \quad \ldots \quad (3)$$

We can take 0·015 m. as an average value of f.

For hypermetropic eyes

$$l-a=f \text{ and consequently } \frac{1}{L}-\frac{1}{A}=f$$

* The refraction of the eye referred to the first principal point is described by Gleichen as "Principal Point Refraction," whilst if referred to the first focal point it is termed "Spectacle Refraction." English ophthalmic writers describe them respectively as "True and Nominal Refractions." As will be seen in the text, the Nominal Spectacle, or L-refraction is numerically equal to the power of the distance-correcting glass, with sign changed. Myopia, therefore, which is corrected by a negative glass has a positive refraction and may be regarded as excess of refracting power. The reverse is the case in hypermetropia.—Trans.

This equation agrees with the previous one for the myopic eye, when we give negative signs to a and l, and hence to A and L, which is correct; since in the case of the hypermetropic eye the far point, which is the object point, lies to the *right* of the reference points.

Ex. 57.

A myopic eye has spectacle refraction $L=8Dp.$ What is the ordinary refraction A?

In equation (3) put $L=8$ and $f=0{\cdot}015$ m.

$$\frac{1}{A}-\frac{1}{8}=0{\cdot}015 \text{ or } A=7{\cdot}14.$$

Ex. 58.

For a myopic eye $A=5{\cdot}5$. What is L?

$$\frac{1}{5{\cdot}5}-\frac{1}{L}=0{\cdot}015 \text{ or } L=6\ Dp.$$

Ex. 59.

The focal point refraction of a hypermetropic eye is $L=-5\ Dp.$ What is the value of the principal point refraction, A?

$$\frac{1}{A}-\left(-\frac{1}{5}\right)=0{\cdot}015 \text{ or } A=-5{\cdot}4\ Dp.$$

The following table gives corresponding values of L and A in dioptres.

Myopia.		Hypermetropia.		Myopia.		Hypermetropia.	
A	L	A	L	A	L	A	L
0·99	1	1·01	1	9·43	11	13·17	11
1·94	2	2·06	2	10·17	12	14·63	12
2·87	3	3·14	3	10·86	13	16·15	13
3·77	4	4·25	4	11·57	14	17·72	14
4·65	5	5·40	5	12·25	15	19·35	15
5·50	6	6·59	6	12·90	16	21·05	16
6·33	7	7·82	7	13·55	17	22·82	17
7·14	8	9·09	8	14·18	18	24·66	18
7·92	9	10·40	9	14·79	19	26·57	19
8·69	10	11·76	10	15·38	20	28·57	20

56. Change of Refraction through Spectacle Lenses.

If, in Fig. 52, P be the far point of a myopic eye, or in general, a point on which the eye is sharply focussed, and if we require that the eye see sharply some other point P_1 without altering the refraction condition of the eye, then we must place in front of it a spectacle glass. This glass must

be so placed that rays from the point P_1 must appear to come, after refraction through it, as if they originated at the point P. Under these circumstances no change has taken place in the ray paths, as far as the eye is concerned. If the point P_1, on which the aided eye is focussed, lie to the right of P, *i.e.* nearer to the eye than P, the rays leaving P_1 have to be made less divergent; that is, a convex glass must be used. Conversely, if P_1 lie farther from the eye than P, a concave glass is necessary. Calling the vertex of the glass (assumed infinitely thin) S, and the first principal point of the eye H, we can put

$$PH=a \qquad PS=l \qquad P_1H=a_1 \qquad P_1S=l_1$$

also ϕ is the focal length of the glass. P is the image of P_1 and both lie on the same side of the glass, hence

$$\frac{1}{l_1}-\frac{1}{l}=\frac{1}{\phi}$$

put

$$\frac{1}{\phi}=D_0 \qquad \frac{1}{l_1}=L_1 \quad \text{and} \quad \frac{1}{l}=L$$

then

$$D_0=\frac{1}{l_1}-\frac{1}{l}=L_1-L \qquad \dots \qquad .. \qquad (4)$$

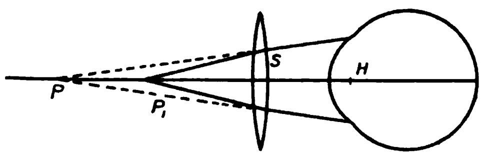

FIG. 52.

In words:

The power D_0 of the correcting glass is equal to the required refraction L_1 referred to the spectacle point, minus the refraction L of the unaided eye, also referred to the spectacle point.

The refraction is to be taken as positive when the far point or the point of fixation of the eye, lies in front of the eye—myopic eye; and negative when that point lies behind the eye—hypermetropic eye.

It must be remembered that

Myopic refractions are positive.
Hypermetropic refractions are negative.

Under these circumstances equation (4) holds for all refraction conditions.

If the desired refraction be zero, so that the eye is corrected for distant objects, then $L_1 = 0$, and

$$D_0 = -L$$

The power of a glass for distance work (distance glass) is equal to the refraction of the eye referred to the spectacle point, with sign changed; that is, the glass is negative for a myopic eye, and positive for a hypermetropic eye.

Glasses are comfortably carried about 14 to 15 mm. in front of the eye, almost the same distance that the first focal point is away. This latter point is therefore considered as the spectacle point (S in Fig. 52), and the focal point-refraction is termed the spectacle refraction. Consequently we have the rule:

The focal point refraction L is equal to the power of the distance glass.

As has been seen above, the focal point refraction L may be calculated from the usual refraction A, or may be obtained from the table in § 55.

Ex. 60.

What is understood by a myope of $8Dp.$?

If it be not stated whether the refraction is reckoned from the focal point or from the principal point, the latter must be assumed. Hence $A = 8$. The distance of the far point H gives $a = \frac{1}{8}$ m $=$ 12·5 cm, therefore the distance of this point from F is $l = 12{\cdot}5 - 1{\cdot}5 = 11$ cm $= 0{\cdot}11$ m. Thus $L = \frac{1}{0\cdot 11} = 9\frac{1}{11}$ $Dp.$ reckoned from the first focal point. Hence the glass necessary to correct for distance vision must have a strength of $-9\frac{1}{11}$ $Dp.$

Ex. 61.

What distance glass must be employed by a hypermetrope of 8 $Dp.$?

Since $A = -8$, $a = \frac{1}{A} = -\frac{1}{8}$m $= -12{\cdot}5$ cm.; the far point thus lies 12·5 cm. from H behind the eye or 14 cm. from F. Therefore $L = -\frac{1}{0\cdot 14} = -7\frac{1}{7}Dp.$ is the focal point refraction and the distance glass will be $7\frac{1}{7}$ $Dp.$ strong.

Ex. 62.

It is required to move the far point of a myope from 40 cm. to 25 cm. away. What glass must be employed?

$$L = \frac{1}{0\cdot 4} \text{ and } L_1 = \frac{1}{0\cdot 25}$$

Hence from equation (4)

$$D_0 = L_1 - L = 4 - 2\cdot 5 = 1\cdot 5\ Dp.$$

Ex. 63.

The distance glass of a hypermetrope is 5 *Dp*. What glass must be used, in order that his far point lies 30 cm. from *F*?

$$L=-5,\ L_1=\frac{1}{0\cdot 3}=3\tfrac{1}{3}\ Dp.$$

$$\therefore\ D_0=3\tfrac{1}{3}-(-5)=8\tfrac{1}{3}\ Dp.$$

Ex. 64.

A myope, whose far point distance is 20 cm. from *F*, is provided with a glass of -4 *Dp*. At what distance from *F* does the aided eye perceive sharply?

$$L=5 \text{ and } D_0=-4$$

Hence from equation (4)

$$L_1-5=-4 \text{ or } L_1=\frac{1}{l_1}=1$$

i.e.,

$$l_1=1 \text{ m.}$$

Ex. 65.

A myope has his far point distance $l=10$ cm., and is provided with a glass of $+5$ *Dp*. Where does his far point lie with such a glass?

$$D_0=5\ ;\ L=\frac{1}{0\cdot 1}=10.$$

$$\text{Hence } 5=L_1-10 \text{ or } L_1=15. \quad \text{Hence } l_1=\frac{1}{15} \text{ m.} =6\tfrac{2}{3} \text{ cm.}$$

The far point is therefore $6\frac{2}{3}$ cm. from the first focal point of the eye.

Ex. 66.

A hypermetrope has a distance glass of 10 *Dp*. Where will his far point lie when he uses a glass of 15 *Dp*.?

$$D_0=15 \text{ and } L=-10. \quad \text{Also } 15=L_1+10$$

or

$$L_1=\frac{1}{l_1}=5 \text{ and } l_1=20 \text{ cm.}$$

57. Combination of the Eye and Spectacle Lens.

If a spectacle glass (thin lens) of power D_0 be placed in front of an eye of power D and focal distances f and f', the two together form a centred system. The first and second principal points of the eye will be designated as usual by H and H', the principal point of the lens by S (*see* Fig. 52) which is at the vertex. In order to find the power D_c of this combination and the positions of its principal points, we resort to the general equations X. and XI., § 33. Since the distance between the two adjacent principal points of

this combination is practically wholly in air, the distance SH suffers no reduction in accordance with Gullstrand's method.

In Fig. 52 we have

$$SH = d$$

hence

$$D_c = D_0 + D - dD_0D \quad \dots \quad \dots \quad (5)$$

and

$$h = -\frac{dD}{D_c} \quad \dots \quad \dots \quad (6)$$

$$h' = -\frac{nd\,D_0}{D_c} \quad \dots \quad \dots \quad (7)$$

It will be observed that since h, the distance of the first principal point of the combination from the first principal point S of the first system, lies in air, no reduction is necessary; whilst h', the distance of the second principal point of the combination from the second principal point of the second system, being in the vitreous humour of refractive index n, it is necessary to reduce by dividing by n, as seen in equation (7).

Ex. 67.

A spectacle glass has a power $D_0 = 10\,Dp.$ and $d = 30$ mm. $= 0{\cdot}03$ m. What is the power of the combination and where do the principal points lie if the power of the eye is $D = 64\ Dp.$?

$$D_c = 10 + 64 - 0{\cdot}03 \times 10 \times 64 = 54{\cdot}8\,Dp.$$

Further

$$h = -\frac{0{\cdot}03 \times 64}{54{\cdot}8} = -0{\cdot}035\text{m.};\ h' = -\frac{1{\cdot}3365 \times 0{\cdot}03 \times 10}{54{\cdot}8} = -0{\cdot}0073\text{m.}$$

The first principal point lies 35 mm. from S towards the *right* (taking the negative sign into consideration) and is 5 mm. to the right of H since $HS = d = 30$ mm. The second principal point lies 7·3 mm. to the left of H'.

A particular case occurs when the glass is situated at the first focal point of the eye. In this case:

$$d = f = \frac{1}{D}$$

and equations (5), (6), and (7) become

$$D_c = D \quad \dots \quad \dots \quad \dots \quad (8)$$

$$h = -\frac{1}{D} = -f \quad \dots \quad \dots \quad (9)$$

$$h' = -\frac{nD_0}{D^2} \quad \dots \quad \dots \quad \dots \quad (10)$$

The equation $D_c = D$ states that the power of the combination is equal to that of the eye; and since $D = \frac{1}{f} = \frac{n}{f'}$, the focal distances of the combination are also equal to those of the eye.

The equation $h = -f$ signifies that the distance from S (Fig. 52) of the first principal point of the combination is f towards the right. Evidently this distance reaches the first principal point of the eye, so that the latter point is at the same time the first principal point of the combination. Since $\frac{n}{D_2} = ff'$, equation (10) may be written

$$h' = -\frac{ff'}{\phi} \quad \dots \quad \dots \quad (11)$$

where $\phi = \frac{1}{D_0} =$ the focal length of the glass.

Since f and f' are positive, h' depends solely on the sign of ϕ and is consequently negative for convex glasses, positive for concave. A negative sign for h' shows that the second principal point of the combination is displaced to the left, that is towards the front, by an amount $\frac{ff'}{\phi}$; this occurs with convex glasses. By the introduction of a concave glass this relation is reversed. In every system the first principal point and the first nodal point are distant f and f' respectively from the first focal point; and the second principal point and the second nodal point are distant f' and f respectively from the second focal point. Hence, since in the case we are considering, f and f' are identical with the focal distances of the eye, it is seen that the first cardinal points (F, H, K) of the combination remain unaltered in position; whilst the second cardinal points (F', H', K') suffer equal displacements of $\frac{ff'}{\phi}$.

Hence we have the following rule;

If a correcting glass, whose thickness is negligible, be placed at the first focal point of the eye, then the focal distances of the combined system equal those of the unaided eye. The front (i.e., *the first*) *cardinal points of the combination are coincident with those of the unaided eye; the back* (i.e., *the second*) *cardinal points are displaced by equal amounts, which latter are numerically equal to the product of the two focal distances*

of the eye divided by the focal length of the correcting glass. For convex glasses the displacement is towards the front and for concave towards the back of the eye.

Ex. 68.

What are the displacements of the back cardinal points of the eye when a glass of $\pm 5Dp.$ is placed at the first focal plane of the eye?

In equation (11) for the displacement of the second principal point, putting $f=15{\cdot}5$ mm., $f'=20{\cdot}7$ mm. and $\phi=200$ mm, then

$$h' = \mp \frac{15\cdot 5 \times 20\cdot 7}{200} = \mp 1\cdot 6 \text{ mm.}$$

The displacement of the other cardinal points is of the same amount.

58. Size of the Retinal Image for the Aided Ametropic Eye.

If an ametropic eye be corrected for distance by a glass of power D_0, the second focal point of the combination coincides with the retinal point. A distant object whose visual angle is u forms an image whose size is given by $y''=\frac{u}{D_c}$ (see equation (20) § 22). The size of the image formed by an emmetropic eye of the same power D as the ametropic eye would be $y'=\frac{u}{D}$. The ratio $\frac{y''}{y'}=K$ is known as the **spectacle magnification.** Then $K=\frac{D}{D_c}$.

Equation (5) may be written

$$\frac{D_c}{D} = 1 - D_o\left(d - \frac{1}{D}\right)$$

Here $\frac{1}{D}$ is the first focal distance of the eye, $\left(d-\frac{1}{D}\right)$ is the distance x of the glass from the first focal point of the eye.

Thus

$$\frac{D_c}{D} = 1 - xD_o$$

or

$$K = \frac{1}{1-xD_o} \qquad \dots \qquad \dots \qquad (11a)$$

Ex. 69.

What influence has the distance glass on the size of the retinal image of a myope?

Here D_0 is negative and x is positive if the glass lie outside the focal distance of the eye. The product xD_0 is consequently negative. Since $-xD_0$ is positive, the denominator of the expression for K is greater than unity and therefore K is less than 1. In this case there occurs a diminution in size of the retinal image as compared with that formed by the emmetropic eye. If the glass be situated within the focal distance of the eye, then x is negative and the product xD_0 is positive. The denominator in the expression for K becomes less than 1, and K becomes greater than 1. Compared with the emmetropic eye the retinal image is thus magnified.

Ex. 70.

What influence has the distance glass on the size of the retinal image of a hypermetrope?

By analagous reasoning the glass magnifies when situated outside the focal distance of the eye, and diminishes the size when inside the focal distance.

Ex. 71.

A myope is provided with a correcting glass of $-8Dp.$ situated 5 mm. from the focal point of the eye. What is the reduction in size of the retinal image?

$$D_0=-8 \text{ and } x=0\cdot005 \text{ m.}$$

Hence

$$K=\frac{1}{1+8\times0\cdot005}=\frac{1}{1\cdot04}=0\cdot96.$$

The diminution in size is therefore about 4 %.

Ex. 72.

What is the magnification when a myope of $10Dp.$ has his correcting glass close to the cornea (called a contact glass)?

In this case

$$x=-13 \text{ mm}; \ D_0=-10$$

$$\therefore K=\frac{1}{1-0\cdot013\times10}=1\cdot15.$$

Ex. 73.

As in Ex. 72 except that $D_0=-20\ Dp.$

In this case

$$K=\frac{1}{1-0\cdot013\times20}=1\cdot35.$$

In the case of strong myopia, a contact glass produces appreciable improvement in vision, assuming that no annoyance, such as irritation of the eye, is caused.

59. Curvature-and Axial-Ametropia.

Ametropia may result from an abnormal change in the power of the eye, due to a large change in the refractive index or in a strong deviation from the normal of the curvatures of the refracting surfaces. This form of

ametropia is comparatively rare. Broadly, the aphakic (or lensless) eye belongs to this class. These forms of ametropia are usually included in the term **Curvature Ametropia.**

More frequently ametropia results from an abnormal elongation (myopia), or shortening (hypermetropia) of the eyeball, whilst the power does not differ to any great extent from that of the emmetropic eye. This form is called **Axial Ametropia.**

In Fig. 53, P is the far point of an axial myopic eye, N the conjugate retinal point, F and F' are the focal points. With our previous nomenclature, $PF=l$, $F'N=l'$, and f, f' and D, the first and second focal distances and power respectively; then

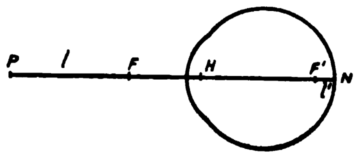

FIG. 53.

$$l\,l' = ff',$$

and for the elongation we have

$$l' = \frac{ff'}{l}.$$

If n be the refractive index of the vitreous humour, then

$$D = \frac{1}{f} = \frac{n}{f'},$$

and since $\frac{1}{l} = L$, and if the "reduced" elongation be $\lambda' = \frac{l'}{n}$ then

$$\lambda' = \frac{L}{D^2} \dots \qquad \dots \qquad \dots \quad (12)$$

Equation (12) applies also to hypermetropia in which case λ' is the reduced shortening; hence the following rule:

The reduced elongation or shortening in an axial ametropic eye is equal to the quotient of the power of the distance glass and the square of the power of the eye.

Ex. 74.

The distance glass of an axial ametrope has a power of 4 *Dp*. What is the magnitude of the elongation or shortening of the eye, if the power of the latter be 64 *Dp*?

In equation (12) substituting $D = 64$ and $L = 4$

$$\lambda' = \frac{4}{64^2} = \frac{1}{1024}\text{ m.}$$

hence $l' = n\lambda' = \frac{1 \cdot 3365}{1024}$ m. $= 1 \cdot 33$ mm.

Ex. 75.

What myopic refraction corresponds to an elongation of 2 mm?

$$l'=0\cdot 002 \text{ m. and } \lambda'=\frac{l'}{n}=\frac{0\cdot 002}{1\cdot 3365}.$$

Hence

$$L=\frac{0\cdot 002\times 64^2}{1\cdot 3365}=6\cdot 13\,Dp.$$

From Table §55, $A=5\cdot 6$ *Dp.**

60. Angle of Vision.

The angle under which an object appears at the eye, called the angle of vision, has been the subject of various discussions. For example, in the case of *near* objects, it becomes doubtful at what point in the neighbourhood of the eye, the vertex of this angle is to be taken. In optical instruments of precision the vertex lies at the rotation point† of the instrument. We will, however, discuss the relations involved in the case of the unaided eye.

In Fig. 54, let $PQ=y$ be an object whose sharp retinal image is y'. Join the point Q to F the first focal point, to H the principal point, to R the centre of the entrance pupil and to K the nodal point. Denote the corresponding angles (see Fig. 54) along the axis by u_f, u_h, u_r and u_k; then from the

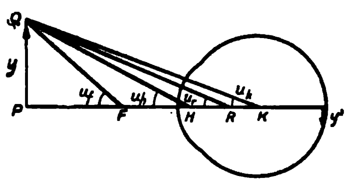

FIG. 54.

* Consider a lens of focal length, say 10 inches, distant 8 inches from a screen. This lens has a power of 4 Dp., wherever it may be placed. For a distant object to be focussed on the screen, we may add a + 1 Dp. lens in contact with it, or we may simply move the 4 Dp. lens 2 inches away from the screen. Thus moving a positive lens forward (away from the screen) is equivalent as far as the screen is concerned, to increasing the power of the lens. This has been described as increasing the "effectivity" of the lens. Obviously the question of variation of effectivity of a lens depends on the sign of the lens, and the relation of the position of the focal point to the position of the screen. If we replace the screen by the eye, and the lens by a spectacle lens, then variations of the position of the lens with respect to the eye result in variations of "effectivity." Care must be exercised to distinguish between this variation of effectivity and the actual changes in the resultant combination of the eye and spectacle. The foregoing §57 is of fundamental importance in ophthalmic optics. The results summarised therein deal more especially with the action of the spectacle lens when situated at the first focal point of the eye. They may, however, be extended by introducing Eq. 12 [§59] to deal with any refractive condition of the eye, and with any position of the lens. At the same time similar considerations as may be formed in §58 yield the sizes of images produced by the combined system of eye and spectacle. An investigation on such lines shows the extreme importance of considering the variation in the positions of the second cardinal points of the system and the dangers of considering only the variation of effectivity (*see* also and compare "True Action of Lenses in Ametropia," by Lionel Laurance, Institute of Ophthalmic Opticians, Nov. 1919. Optician and Scientific Instrument Maker. Nov. 1919.)—Trans.

† *i.e.*, the point round which the instrument is rotated in order to bring the two extremities of the object successively on the intersection of the crosswires.—Trans.

series of triangles PQF, PQH, PQR and PQK, when the angle u is assumed *small*, we have:

$$u_f = \frac{y}{l} \quad u_h = \frac{y}{a} \quad u_r = \frac{y}{p} \quad u_k = \frac{y}{b} \qquad \ldots \quad (13)$$

From the relation $\frac{y'}{y} = \frac{f}{l}$ we obtain from the first of the equations (13)

$$y' = f\,u_f \ldots \qquad \ldots \qquad \ldots \quad (14)$$

Hence the following rule:

The retinal image of an object is equal to the first focal distance of the eye, multiplied by the angle subtended at the first focal point by the object, and holds for any refraction condition of the eye.

So long as f does not change, that is for unaccommodated vision, in the case of axial ametropic eyes, all objects which are seen sharp under the same angle at F have their retinal images equal in size.

From equation (8) § 11,

$$\frac{y'}{y} = \frac{a'}{na}$$

where n is the refractive index of the vitreous humour. The second of equations (13) therefore becomes

$$y' = u_h \cdot \frac{a'}{n} \qquad \ldots \qquad \ldots \quad (15)$$

giving the rule;

The retinal image of an object is equal to the reduced distance of the second principal point from the fovea, multiplied by the angle which the object subtends at the first principal point.

The magnitude of a' varies for different axial ametropic eyes, since it depends on the length of the eye. When, however, we make the justifiable assumption that the second principal point is not displaced by accommodation, a' remains constant during accommodation, and we can state the rule:

For every accommodation condition, the same eye produces retinal images which are the same size for all objects which subtend equal angles at the first principal point.

Further, employing the equation (*see* equations (4) § 39)

$$\frac{y'}{y} = \frac{b'}{b} = \frac{p'}{np} \cdot \frac{\rho}{\rho'}$$

where n is the refractive index of the vitreous humour, the two remaining equations (13) give:

$$y' = u_r . p' . \frac{\rho}{n\rho'} \quad \ldots \quad \ldots \quad (16)$$

$$y' = u_k\, b' \ldots \quad \ldots \quad \ldots \quad (17)$$

in which the size of the retinal image is expressed in terms of the angle in the object space subtended by the object either at the pupil centre or at the nodal point.

Since the magnitudes y' in equations (14), (15), (16), (17) have the same value, the relations between the angles u_f, u_h, u_r, u_k may be obtained by equating these expressions.

For very distant objects these angles become equal to one another, and then

$$f = \frac{a'}{n} = \frac{p'\rho}{n\rho'} = b'$$

expressing simply the condition for emmetropia (*cf.* § 54).*

61. Accommodation.

By the term "Accommodation" is understood that characteristic quality of the eye, by virtue of which it is able to alter its refraction and thus focus sharply objects at different distances from the eye. The alteration in the refraction of the eye is brought about by a change in the form of the lens. The anterior vertex of the lens (S_2 in Fig. 48) is displaced towards the cornea by an amount of about 0·4 mm., corresponding to a change of refraction of 7 *Dp*. The anterior radius of the lens (r_2) is shortened at the same time by an amount of about 4 mm. The position of the posterior vertex of the lens (S_3) does not appreciably alter, whilst the radius on the average is only shortened by 0·5 to 0·8 mm. (*see* the table of the accommodated schematic eye of Helmholtz, § 53).

Of the other effects entering into accommodation, we will mention here simply the contraction of the pupil and the falling down of the crystalline lens according as the face

* Attention may well be drawn at this point to an important refraction condition of the eye, omitted in the text, in which the ocular refraction is different in different meridians, *i.e.* astigmatism. Considering astigmatism of a regular type only, the main cause usually lies in the differing curvatures of the cornea in different meridians giving rise to astigmatism of the first type (*see* §44). Owing to the fact that the crystalline lens is not accurately centred, a slight amount of oblique astigmatism is introduced (§46). The optical correction, as in ametropia consists in obtaining single retinal focus for distant objects, by means of sph. cyl. or toric lenses.—Trans.

is turned down, or is directed upwards (looseness of the lens). The displacement of the lens on this account amounts to about 0·25 to 0·3 mm. When the lens has its lowest power (condition of rest of the eye) the eye is said to be focussed on the **Far Point.*** On the other hand, when it has the greatest power and consequently greatest convexity, the eye is said to be focussed on the **Near Point.†** For an emmetropic eye the far point lies at infinity and the near point at a finite distance in front of the eye. In myopia both the far and near points lie at finite distances in front of the eye; whilst in the case of hypermetropia, in which the far point is virtual and lies behind the eye, the near point may either lie virtually behind the eye, or may actually exist as a real point in front of the eye.

In Fig. 55, H and H' are the first and second principal points of the eye, F the first focal point, P_2 the far point, and P_1 the near point. **By the act of accommodation the alteration of position of the principal points is extremely small** and consequently may generally be neglected. By referring to the table of the schematic eye (§ 53) it will be seen that for an alteration of focus of the eye from infinity to a point 140·3 mm. from the vertex of the cornea, the first principal point is displaced only 0·11 mm., and the second principal point 0·16 mm. For this reason the principal points may be conveniently employed as points of reference.

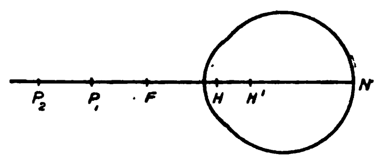

FIG. 55.

If we state

Distance of far point P_2 from $H = a_2$

Distance of near point P_1 from $H = a_1$

then $(a_2 - a_1)$ is called the **Range of accommodation.**

Further:

The refraction of the eye when focussed on P_2 is $\frac{1}{a_2} = A$

" " " " " " " " P_1 is $\frac{1}{a_1} = A_1$

The difference between the refractions of the eye when

* Punctum remotum or P.R.—Trans.

† Punctum proximum, or P.P.—Trans.

focussed on the far and near points respectively, is termed the **Amplitude of Accommodation** and will be designated by A_{KK}.

Hence:

$$A_{KK}=A_1-A_2 \quad \dots \quad \dots \quad \dots \quad \dots \quad (18)$$

It will thus be seen from this equation that the measure of accommodation is equal to the power of an infinitely thin lens, which, when placed at the first principal point, images the far point at the near point. If, in addition, in Fig. 55, N is the point where the optic axis cuts the retina, then N must be conjugate to both P_2 and P_1 respectively. If A_2' and A_1' represent the vergences in the image space corresponding to A_2 and A_1 respectively, we have:

$$A_2+A_2'=D_2$$
$$A_1+A_1'=D_1$$

where D_2 is the power of the eye at rest and D_1 that of the accommodated eye.

Since the change in position of the second principal point during accommodation may be neglected, $A_2'=A_1'$, and by subtraction, from the two equations above

$$A_1-A_2=A_{KK}=D_1-D_2 \dots \quad \dots \quad \dots \quad (19)$$

The Amplitude of Accommodation is equal to the change in power of the eye.

From equation (19)

$$D_1=D_2+A_1-A_2 \quad \dots \quad \dots \quad \dots \quad (20)$$

where D_1 represents (approximately) the refraction of the accommodated eye.

From this last equation we can determine in a simple manner the alteration in power of the eye for any condition or phase of accommodation.

Ex. 76.

A myope of 7 *Dp.* exerts an accommodation of 3*Dp.* If the power of the eye at rest be 64 *Dp.*, what is the power of the accommodated eye?

$$A_1-A_2=3 \text{ and } D_2=64$$

Hence

$$D_1=67\,Dp.$$

Ex. 77.

A myope, having, in the condition of rest, first focal distance $= 15 \cdot 5$ mm. and the distance of the far point $a_2 = 15$ cm., is accommodated to 8 cm. What is the focal distance f_1 in the accommodated condition?

We have

$$A_2 = \frac{100}{15} : A_1 = \frac{100}{8} ; A_1 - A_2 = 5 \cdot 8$$

Now

$$D_1 = \frac{1}{f_1} = \frac{10000}{155} + 5 \cdot 8 = 64 \cdot 5 + 5 \cdot 8 = 70 \cdot 3.$$

Therefore

$$f_1 = \frac{1}{70 \cdot 3} \text{m} = 14 \cdot 2 \text{ mm}.$$

62. Emmetropic and Ametropic Presbyopia.

With increasing old age the eye lens becomes harder with a consequent decrease in accommodative power; the near point recedes from the eye, whilst the far point, up to ages of about 55 years, does not sensibly alter. With older age the position of the far point also alters in the sense of diminishing refraction, so that the emmetropic eye becomes slightly long-sighted. This phenomenon is called **Presbyopia.**

The following table, based on mean values for an emmetropic eye, shows conveniently, according to Donders, the decline in accommodation with increasing old age. (*See* Table below.)

In hypermetropia there are three cases to be distinguished; for whilst the far point lies always virtually behind the eye, the near point may be also virtual, or infinitely distant, or real at a finite distance in front of the eye.

With Hypermetropia as with Myopia accommodation brings the point of sharp focus (that is the point whose image lies exactly on the retina) nearer to the vertex of cornea; only in hypermetropia this takes place through infinity. In the act of accommodation the point of sharp focus moves away from the far point, within the virtual region, towards infinity. The distance from the first principal point becomes greater, *i.e.*, the near point distance a_1, which has a negative value, is really, in its absolute value, greater than the far point distance a_2. By more extended efforts of accommodation, the distance a_1 may become infinitely great, or may even go still further and acquire a positive value.

Emmetropic Eye.

Distances are measured from the first principal point of the eye.

Age in Years.	Distance of near point in cms.	Distance of far point in cms.	A_{KK}. Amplitude of Accommodation in dioptres.	L_{KK}.
10	7·1	∞	14	17·9
15	8·3	∞	12	14·7
20	10	∞	10	11·7
25	12·8	∞	7·8	8·8
30	14·3	∞	7	7·8
35	18·2	∞	5·5	5·9
40	22·2	∞	4·5	4·8
45	28·6	∞	3·5	3·6
50	40	∞	2·5	2·6
55	66·6	− 400 (H* = 0·25)	1·75	1·79
60	200	− 200 (H = 0·5)	1	1
65	− 400	− 133 (H = 0·75)	0·5	0·5
70	− 100	− 80 (H = 1·25)	0·25	0·25
75	− 57·1	− 57·1 (H = 1·75)	0	0
80	− 40	− 40 (H = 2·5)	0	0

The amplitude of accommodation is equal to the change in power of the whole eye (equation (19)). The latter is essentially the same in emmetropia as in axial ametropia, which arises merely from an elongation or shortening along the axis of the eye; hence the first and last columns in the table hold for all degrees of myopia and hypermetropia, whilst, of course, the distances of the far and near points and consequently the region of accommodation, for ametropic eyes, suffer considerable alterations.

Ex. 78.

For an emmetropic eye, the near point lies 14·3 cm. from the first principal point of the eye. What is the amplitude of accommodation and also the range?

Since the far point lies at infinity, the accommodation range is also infinite.

Further

$$A_{KK} = \frac{100}{14 \cdot 3} = 7\ Dp.$$

* H, in the table is a contraction for Hypermetropia.

Ex. 79.

A myope has far point distance $a_2 = 25$ cm. and near point distance $a_1 = 12$ cm. What are the magnitudes of the amplitude and range of accommodation ?

$$A_1 = \frac{1}{a_1} = \frac{1}{0\cdot12} = \frac{100}{12} = 8\tfrac{1}{3} \; ; \; A_2 = \frac{1}{a_2} = 4 \; ;$$

hence $A_{KK} = 4\frac{1}{3}$ $Dp.$ The range of accommodation is 13 cm.

Ex. 80.

A hypermetrope has far point distance $a_2 = -100$ cm. (behind the eye), and near point distance $a_1 = 50$ cm. (in front). What are the magnitudes of the amplitude and range of accommodation ?

The range of accommodation is here infinite since it includes infinity.

Further

$$A_1 = \frac{100}{50} = 2 \; ; \; A_2 = -\frac{100}{100} = -1.$$

$$A_{KK} = A_1 - A_2 = 2 - (-1) = 2 + 1 = 3 \; Dp.$$

Ex. 81.

A myope, 30 years old, requires distance glasses of -4 $Dp.$ How far away is the near point ?

From the Table §62, $A_{KK} = 7$ $Dp.$ Since $L = 4$, then $A = 3\cdot77$ $Dp.$

$$\therefore A_1 = A_{KK} + A = 7 + 3\cdot77 = 10\cdot77 \; Dp.$$

$$\text{or } a_1 = \frac{1}{10\cdot77} \text{ m.} = 9\cdot3 \text{ cm.}$$

Ex. 82.

A hypermetrope, 15 years old, requires a distance glass $+2$ $Dp.$ strong. Where does the near point lie ?

From Table §62, $A_{KK} = 12$ $Dp.$

Since $L = -2$, then $A = -2\cdot06$ $Dp.$

$$A_{KK} + A = 12 - 2\cdot06 = 9\cdot94 \; Dp.$$

$$\text{or } a_1 = \frac{1}{9\cdot94} \text{ m.} = 10\cdot06 \text{ cm.}$$

Presbyopia affects the ametropic and emmetropic refraction conditions to equal extents, and is independent of the refraction condition ; nevertheless it appears to affect them quite differently. In emmetropes, the moving away or recession of the near point to 40 cm. between the ages of 45 and 50 years, becomes, on account of the diminution of the size of the retinal image, disagreeably noticeable ; consequently the reading of small print, or fine handwork, presents difficulties. In the case of myopic eyes this effect does not assert itself to the same extent because the near point lies comparatively close to the eye, and its retreat is frequently not perceptible. Myopes of 3 to 4 $Dp.$, whose far-point

distance amounts to about 23–25 cm., are in the fortunate position, even when accommodation is practically eliminated, of being able to read at the ordinary distance away.

In hypermetropia, the recession of the near point—at least in strong hypermetropia and in great age—is mostly not very perceptible, since, usually, the hypermetrope has already been supplied in his youth with a correcting glass for reading.

63. First Focal Point as Origin for Calculation. Amplitude of Accommodation of the Aided Eye.

Up to the present we have taken the first principal point as origin for the measurement of the distances of the far and near points. During accommodation the position of the first principal point varies by little more than $\frac{1}{10}$ mm. and consequently may be assumed as stationary.

It is intended now to utilize the first focal point as point of reference. Although the latter suffers displacement during accommodation, it will be shown below that this displacement is small, and can usually be neglected for practical purposes. Calculations based on intercepts measured from the first focal point permit of considerable simplification, especially in the case of the *aided* eye.

In order to demonstrate the justification for neglecting the displacement of the first focal point, consider the schematic eye; from the table, § 53, we see that the first focal point is displaced about 1·6 mm. towards the cornea, namely, from 13·74 to 12·13 mm., by an exertion of accommodation of about 7 *Dp*. The distance from the first principal point, which moves only about 0·1 mm., thus changes from 15·5 mm. to 14 mm.

In what follows it will be assumed that the first focal point lies 15 mm. from the first principal point. Neglecting the displacement of the first focal point during accommodation, the maximum error introduced amounts to about 1 mm., and only becomes noticeable when dealing with very great refraction values of the eye.

If F be the first focal point of the eye, then from § 34

Distance of the far point P_2 from $F = l_2$

Distance of the near point P_1 from $F = l_1$

$$\text{Focal point refraction for } P_2 = L_2 = \frac{1}{l_2}$$

$$\text{Focal point refraction for } P_1 = L_1 = \frac{1}{l_1}$$

Designating the amplitude of accommodation referred to the first focal point by L_{KK}, we have:

$$L_{KK}=L_1-L_2=\frac{1}{l_1}-\frac{1}{l_2} \quad \dots \quad \dots \quad \dots \quad (21)$$

If the eye be provided with a glass of power D_0, placed at the first focal point, and neglecting the variation of the position of this point during accommodation, the far and near points, by accommodation, are distant l_2' and l_1' from the first focal point respectively. Then the refraction $L'_2=\frac{1}{l_2'}$ and $L'_1=\frac{1}{l_1'}$.

The amplitude of accommodation L'_{KK} is therefore

$$L'_{KK}=L'_1-L'_2.$$

The action of the glass of power D_0 is to shift the far point so that its distance from the vertex of the glass changes from l_2 to l_2'.

Hence:

$$\left.\begin{array}{l} D_0=\frac{1}{l_2'}-\frac{1}{l_2}. \quad \text{Similarly } D_0=\frac{1}{l_1'}-\frac{1}{l_1} \\ \text{or} \\ D_0=L'_2-L_2 \quad \text{and} \quad D_0=L'_1-L_1 \end{array}\right\} \quad \dots \quad (21a)$$

From the last two equations it follows

$$L'_1-L'_2=L_1-L_2\,;\ \textit{i.e.}\ L'_{KK}=L_{KK}$$

The amplitude of accommodation referred to the first focal point remains unaltered with the introduction of spectacle glasses.

If we put

$$\frac{1}{l_1}=L_{KK}+L_2$$

then

$$D_0=\frac{1}{l_1'}-L_{KK}-L_2$$

or, putting the *near point refraction of the aided eye*, $\frac{1}{l_1'}=L'_1$

$$D_0=L'_1-L_{KK}-L_2 \quad \dots \quad (22)$$

To obtain a near point refraction of L_1', for an eye whose amplitude of accommodation is L and whose distance glass has power L_2, a spectacle of strength $L_1'-L_{KK}-L_2$ must be used.

The magnitudes L_{KK} for various ages are given in the table, § 62.

Equation (22) is remarkably simple compared with the relations obtained when the various distances are referred to the first principal point.

If, by using a spectacle lens, it be desired to make the near point lie 25 cm. from F, then $l_1'=0{\cdot}25$ m., $L_1'=4$, and equation (22) becomes

$$D_0=4-L_{KK}-L_2$$

Here again it should be noted that the refraction L is

positive for **myopes**

and **negative** for **hypermetropes.**

Ex. 83.

A hypermetrope of 60 years employs a distance glass of 4 *Dp*. What glass will he require to bring his near point 25 cm. from F?

From Table §62 $L_{KK}=1$; $L_2=-4$ and therefore $D_0=4-1+4=7\,Dp.$

Ex. 84.

A hypermetrope of 20 years employs a distance glass of 8 *Dp*. What glass will be required to make the near point lie 30 cm. from the first focal point of the eye?

From the Table §62, $L_{KK}=11{\cdot}7$. Further $L_1'=\frac{10}{3}=3{\cdot}3$ and $L_2=-8$.

Therefore $D_0=3{\cdot}3-11{\cdot}7+8=-0{\cdot}4$. Therefore the glass is concave and 0·4 *Dp*. strong.

Ex. 85.

A myope, 60 years old, whose spectacle refraction $L_2=10$ *Dp*., requires his near point to lie 25 cm. from F. What glass must he employ?

From § 62

$$L_{KK}=1.$$

Hence

$$D_0=4-1-10=-7\,Dp.$$

Ex. 86.

A myope has his far and near point distances $l_2=15$ cm., $l_1=10$ cm. Where do these points lie when he employs a glass of -3 *Dp.*?

From equation (21*a*)

$$-3=\frac{1}{l_2'}-\frac{1}{0{\cdot}15} \qquad -3=\frac{1}{l_1'}-\frac{1}{0{\cdot}1}$$

$$\therefore l_2'=27\tfrac{3}{11}\text{ cm. and } l_1'=14{\cdot}3\text{ cm.}$$

Ex. 87.

The near point distance of a hypermetrope is $l_1=40$ cm. Where does his near point lie when the eye is corrected with a distance glass of 4 *Dp.*?

Here $D_0=4$, $l_1=0{\cdot}4$ m. and from equation (21a)

$$4=\frac{1}{l_1'}-\frac{1}{0{\cdot}4} \qquad l_1'=\frac{2}{13}\text{m.}=15{\cdot}4\text{ cm.}$$

64. Power of Accommodation of Crystalline Lens.

Since we do not know the curvatures or positions of the surfaces of the crystalline lens at each stage of accommodation, it is difficult to establish what amount of alteration of power of the crystalline lens plays an effective part in the accommodative power of the whole eye.

However, under an assumption which corresponds very closely with actual conditions, we can do this in a simple manner.

We have seen already that the positions of the principal points remain approximately unchanged during accommodation, so that these variations of position may be neglected. In addition, we may make the assumption that the displacement of the first principal point of the crystalline lens also may be neglected. In the schematic eye of Helmholtz, which has an accommodative power of about 7 *Dp.*, the distance from the anterior vertex of the lens to the first principal point of the lens is 2·126 mm. in the condition of rest, and 1·987 mm. when accommodated. The distances from the vertex of the cornea then are 5·712 mm. and 5·189 mm., so that the displacement in question amounts to 0·523 mm.

Even if this change be not so small as that of the principal point of the whole eye, yet it is unimportant compared with the change in focal length of the crystalline lens, which, in the schematic eye, amounts to more than 11 mm.

By neglecting the displacement of the first principal point of the crystalline lens, the great advantage is gained that we may put the distance, d, of this point from the vertex of the cornea, constant during the act of accommodation.

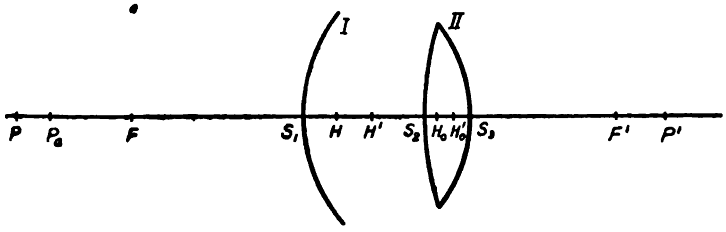

FIG. 56.

In Fig. 56, as heretofore, let S_1 be the vertex of the cornea, H and H' the first and second principal points of the whole eye. H the first principal point of the crystalline lens.

When the eye "fixes" a point P in the object space its power is, say, D; when it is focussed on some nearer point P_a, its power is D_a. The retinal point P', under these circumstances is conjugate both to P and P_a; the power of the crystalline lens in the second case is, of course, increased. In Fig. 56 the points P and P_a are shown comparatively near to the first focal point F, but this is due to lack of space in the page; the actual points would be farther away.

We will now consider the eye as a combination of two systems:

1. The cornea, with common principal point S_1 and power D^I, which is constant during all stages of accommodation.
2. The crystalline lens system with first principal point H_0 and powers D^{II} and D_a^{II} when focussed on the points P and P_a respectively.

Putting the reduced distance between the two adjacent principal points of systems I and II equal to δ, we have

$$\delta = \frac{d}{n} \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

where $d = S_1H_0$ and $n = 1{\cdot}3365$ (refractive index of aqueous humour). When focussed sharply on the point P

$$D = D^I + D^{II} - \delta\, D^I . D^{II} \quad \ldots \quad (24)$$

when focussed sharply on point P_a

$$D_a = D^I + D_a^{II} - \delta\, D^I . D_a^{II} \quad \ldots \quad (25)$$

It should be remembered that the power of the cornea does not change during accommodation. From equations (24) and (25)

$$\begin{aligned} D_a - D &= D_a^{II} - D^{II} - \delta\, D^I\,(D_a^{II} - D^{II}) \\ &= (D_a^{II} - D^{II})\;(1 - \delta\, D^I) \quad \ldots \quad \ldots \quad (26) \end{aligned}$$

We will put the refractions referred to the first principal point equal to $A\left(=\frac{1}{PH}\right)$ and $A_a\left(=\frac{1}{P_aH}\right)$ respectively, then from equation (19) the accommodative power or amplitude of accommodation is

$$A_{KK} = A_a - A = D_a - D \ldots \quad \ldots \quad (27)$$

Putting the accommodative power of the crystalline lens, $D_a^{II} - D^{II} = K_{KK}$, we have from equation (26)

$$A_{KK} = K_{KK}\,(1 - \delta\, D^I) \ldots \quad \ldots \quad (28)$$

The accommodative power of the whole eye is proportional to that of the crystalline lens. As a mean value we may put

$$d = 0{\cdot}0056 \text{ m.}$$

$$i.e., \delta = \frac{0{\cdot}0056}{1{\cdot}3365} = 0{\cdot}0042 \text{ m.}$$

$$D^I = 43\ Dp.$$

so that

$$A_{KK} = 0{\cdot}82\ K_{KK}.$$

If the accommodative power of the crystalline lens be 1 *Dp.*, then that of the whole eye is only 0·82 *Dp.*; or, if the change of refraction of the eye is required to be 1 *Dp.*, then the crystalline lens must have accommodation 1·22 *Dp.*

No particular assumptions were made above with regard to the magnitudes of the powers in question, so that equation (28) is true also for early presbyopia. In fact, in this case the conditions are particularly favourable, since, because of the partial stiffening or lack of resiliency of the substance of the crystalline lens, the first principal point of the latter practically does not move at all. In the same way equation (28) holds for *all refraction conditions.*

65. Entrance Pupil and Exit Pupil of the Eye.

The aperture diaphragm, or simply, the aperture of the human eye is circular and lies on the anterior surface of the crystalline lens. It is called the **Pupil.** The image of this pupil formed by the cornea is the entrance pupil of the eye; the image of it formed in the vitreous humour by the crystalline lens is the exit pupil. The positions of entrance pupil and exit pupil are given in the table §53 for the Helmholtz schematic eye. When focussed on infinity, the magnification at the pupils is given by

$$B_a = \frac{\rho'}{\rho} = 0{\cdot}923.$$

Where ρ = radius of entrance pupil.

ρ' = radius of exit pupil.

The radius of the entrance pupil is 1·131 times the radius of the pupil; and the radius of the exit pupil 1·044 times.*

* For the method of determining the value of B_a, *see* the author's "Einführung in die medizinische Optik." Leipzig, 1904, pp. 77 and onwards.

66. Depth of Focus of the Eye.

If the complete vergence of an eye be Q, the limiting vergences for which sharp image formation still takes place are given by Q_1 in the equation (*cf.* equation (11) in §42) :

$$Q_1 = Q \pm \frac{z_0}{\rho}(\Delta - Q) \quad \dots \quad \dots \quad \dots \quad (29)$$

where z_0 is the radius of those circles of diffusion on the retina which are conceived as points. With regard to the keenness of vision, or as it is called, *visual acuity* (resolving power), the normal eye will conceive as diffusion circles those circles on the retina which correspond to a visual angle in the object space of one minute or more.* If the angle be less than this, then the normal eye will not be able to differentiate the corresponding retinal image from a point.

We find $z_0 = \frac{0{\cdot}0045}{2} = 0{\cdot}0022$ mm.†

where z_0 is half of the corresponding retinal image, Ex. 56.‡

Introducing into equation (29) the refractions referred to the centres of the pupils, that is

$$Q = \rho^2 P \qquad Q_1 = \rho^2 P_1 \qquad \Delta = D\rho\rho'$$

we obtain

$$P_1 = P \pm z_0\left(\frac{D}{\rho} - P\right) \quad \dots \quad \dots \quad (30).$$

An average value of ρ is 2 mm $= 0{\cdot}002$ m. Since z_0 is so small ($z_0 = 0{\cdot}0000022$ m.) we may neglect the term $z_0 P$, so that

$$P_1 = P \pm 0{\cdot}0011\ D.$$

Thus an eye in the refraction condition P, can see sharply all objects within an interval corresponding to an increase or decrease of refraction of $0{\cdot}0011\ D$.

If the eye be focussed on infinity, then $P = 0$, and putting $D = 64$, we have

$$P_1 = \pm 0{\cdot}07.$$

* It has been found that under certain circumstances, *e.g.*, in the case of a fine wire brilliantly illuminated, objects are resolved which subtend an angle of the order 10″. *See also* Chap. XI.—Trans.

† $2z_0 = \omega_0 f = 1$ minute $\times$ 15·5 $=$ 0·00029 $\times$ 15·5 $=$ 0·0045 mm. *See* §109.—Trans.

‡ *See* "Einführung in die medizinische Optik," p. 98.

The value $-0{\cdot}07$ means that the emmetropic eye when focussed on infinity, sees the same as a hypermetropic eye of $+0{\cdot}07$ *Dp*. From the value $+0{\cdot}07$ we obtain

$$p_1 = \frac{1}{P_1} = \frac{1}{0{\cdot}07} = 14 \text{ m.}$$

Thus all objects distant more than 14 m. from the eye appear sharp.

If the visual acuity be S,* differing from unity, z_0 in the above equations must be replaced by $\frac{z_0}{S}$. Thus persons with small S, *i.e.*, persons whose visual acuity is less than normal, command a greater depth of focus.

References.

For a full elementary treatment of the eye, both from the dioptric and ophthalmic standpoints, *see* "Visual Optics and Sight Testing," by Lionel Laurance, published by that author.

* It has been assumed that an eye has normal visual acuity (V.A.) *i.e.*, $S=1$ when it can perceive as separate points, two points whose distance apart subtends an angle of one minute at the eye. All eyes cannot do this. If y_0 be the size of the retinal image of an object just recognizable by the normal eye, and y that of an object just recognizable by a particular eye, then the visual acuity is expressed by $S=\frac{y_0}{y}$.—Trans.

CHAPTER VII.

General Theory of Optical Instruments.

67. Vision through Optical Instruments.

The light taking part in vision with the unaided eye is enclosed within cones of rays proceeding from each point of the object under view, the common base of these cones being the pupil, or, more correctly, the image of the pupil formed by the cornea, *i.e.*, the entrance pupil of the eye. In the case of distant objects these cones of rays from the various points of the object degenerate to cylinders. If the eye move through space, the position of the entrance pupil changes, and the eye receives new sets of cones of rays and, as far as the refraction condition of the eye permits, a sharp image is continually received on the retina. At the same time a very large field of view is embraced, amounting almost to the solid angle of a hèmisphere. This is still essentially true when the eye is provided with a glass, as this governs only the refraction condition and has scarcely any restricting action on the field of vision. It is not, however, true in the case of optical instruments for visual purposes, which usually consist of tubes provided with lenses or reflecting surfaces and with diaphragms. Used with an optical instrument, the entrance pupil of the eye must take up a definite position in order to embrace the greatest possible part of the field of view, and must be directed along a fixed straight line. This line in general is the optical axis of the instrument on which the lenses of the instrument are centred. Further, conditions such as the brightness and sharpness of the image, &c., must be considered; questions which it is usually not necessary to discuss in the case of vision with the unaided eye. In Chapter IV. we have seen that every optical system possesses an effective aperture whose images formed in the object and image spaces respectively by the component systems in front of and behind it, are called the entrance pupil and exit pupil. The function of the entrance pupil of the instrument is the same as that of the entrance pupil of the eye in unaided vision.

It forms the common base for all cones of rays proceeding from the various point objects; these bundles of rays traverse the optical elements of the system, emerging

finally through the exit pupil of the instrument, which is the image of the entrance pupil, into the image space. If now the entrance pupil of the eye coincide with the exit pupil of the instrument, then all these bundles enter the eye, which will see sharply if it be capable of uniting these bundles to points on the retina. If the emerging bundles be cylindrical, the emmetropic unaccommodated eye will see sharply; so also under certain circumstances will the accommodated hypermetropic eye. If the rays of the bundles are diverging, then a sharp image will still be received if the point of divergence be as far away as, or further away than, the near point of the eye. If the bundles of rays are converging, then under certain circumstances a hypermetropic eye will see distinctly. The designer of optical instruments usually tries to arrange that the emerging bundles are cylindrical, so that vision is possible to an emmetrope without accommodation. In addition, means are provided so to alter the vergence of the bundles by an approach or recession of the object relative to the instrument, or by a displacement of the optical elements, that sharp vision is possible also for ametropes. In many cases, as for example, in vision through the Galilean telescope, or in observing the background of the eye by means of the erect-image ophthalmoscope, the exit pupil of the instrument is not accessible (in the latter case the exit pupil of the instrument is also that of the patient's eye) since it does not really exist in the image space. In these cases there is a diminution of the field of view in comparison with that obtained in instruments with real exit pupils. The entrance pupil of an instrument can be found in the following manner according to the general rules given in Chapter IV.: the image in the object space of each diaphragm present, formed by that part of the system lying in front of it, is constructed; then if we wish to image sharply an object plane whose axial point is P, **that aperture image which subtends the smallest angle at *P* is the entrance pupil of the instrument. The image of the latter formed by the whole system is the exit pupil.**

In the design of the instrument the effective aperture, that is the aperture whose image in the object space is the entrance pupil, is made as large as possible, so that the maximum amount of illumination is obtained. This aperture will usually be one of the lens holders, or in certain cases it may even be the wall of the tube. No stop should be inserted anywhere in the system which would limit this amount of illumination.

Apart from their dependence on the magnification required for the instrument, the sizes of the pupils depend principally on the size of the effective aperture; it is possible to increase this, however, only within certain limits, because as the aperture increases the aberrational errors of the instrument rapidly increase. In many instruments, for example, instruments for observing the interior of bodies (cystoscopes, &c.) or periscopes (telescopes frequently many metres long and comparatively small in diameter), the path of the light is necessarily limited to a space which is very long and narrow. Here we have what may be called a "space limitation." Instruments of this kind show most clearly the relations between the position of the object and the positions of the pupils; we will consider a simple case of this kind.

That part of the lens system of the instrument which lies next to the object is generally called the **Objective** and that part of the system next to the eye is the **Eye-piece.**

68. Instruments of two thin Lenses. Positions of entrance and exit pupils.

Fig. 57 shows an optical instrument which consists of two thin lenses, whose vertices are R and S, placed at the ends of a tube $LL_1L_2L_3$. The focal lengths of the lenses are f_1 and f_2; if F and F' are the focal points of the first lens, then $FR=F'R=f_1$. There are no diaphragms in the ordinary sense; the edges of the lenses and the walls of the tube act as such.

In order to find the entrance pupil, we find the image in the object space of the tube $LL_1L_2L_3$; this can be done by the usual method of finding the image produced by a thin lens (§ 28).

If this construction be followed point for point, we find that the image of the part LL', where L' lies perpendicularly over F', is a straight line LM produced to infinity. Similarly the image of the part L_2L'' is the straight line L_2N. GJ is the image of the second lens L_1L_3 produced by the first lens LL_2. The images of the parts $L'L_1$ and $L''L_3$ of the tube are the straight lines TJ and QG stretching to infinity; TJ and QG produced intersect at the first focal point F of the first lens. In this construction the second lens has played no part whatever. If we rotate this figure around the optical axis, we obtain a surface in the object space which is the image of the tube. It consists of two conical surfaces stretching to infinity whose apices are cut

off perpendicular to the axis. The first cone, in this way, has aperture GJ and the second the aperture LL_2. We can conceive these conical images as actually consisting of opaque material and thus obtain a useful representation of the path

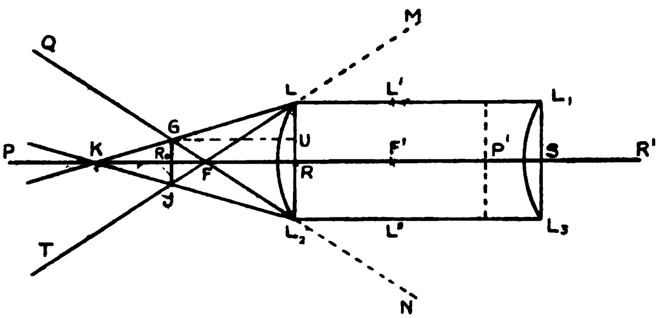

FIG. 57.

of the light through the instrument. To determine which of the two apertures GJ and LL_2, whose semi-diameters are $R_0G=\rho_0$ and $RL=\rho$, is the entrance pupil, we must choose a certain definite position of the object plane which is perpendicular to the axis and cuts the latter in the point P. Join LG and L_2J, produced to cut the axis in K. Since GJ is smaller than LL_2 and in the case considered the tube is a long one, it is evident that for distant object points GJ is the entrance pupil of the instrument. If now the point P approach the instrument along the optical axis until it reach the point K, then both JG and LL_2 subtend the same angle at this point; thus both apertures act as the entrance pupil. If the object point approach still nearer the instrument, LL_2 subtends the smaller angle and hence is the entrance pupil. When the object point P reaches F, both GJ and LL_2 again are the entrance pupils; GJ of course lies behind the object plane, but optically this makes no difference. For any position of the object point still nearer the instrument than F, GJ subtends the smaller angle and is consequently the entrance pupil of the instrument.

Hence we can state:

The front aperture of the tube LL_2 is the entrance pupil of the instrument only when the object point lies between the points K and F; in all other cases the image GJ of the back aperture L_1L_3 of the tube formed by the first lens is the entrance pupil.

The straight line KF is called the **critical distance.**

The exit pupil is always the image of the entrance pupil formed by the whole system. If GJ be the entrance pupil,

then L_1L_3 is the exit pupil; if LL_2 be the entrance pupil, then the exit pupil is the image of LL_2 formed by the second lens of focal length f_2.

To find the position and length of the critical distance we proceed as follows:

Draw GU parallel to the axis; then $LU=\rho-\rho_0$. From the similar triangles KR_0G, GUL, and KRL:

$$\frac{KR}{\rho}=\frac{KR_0}{\rho_0}=\frac{GU}{\rho-\rho_0}=\frac{R_0F+FR}{\rho-\rho_0}$$

Put $F'S=e$; then since R_0 and S are conjugate points,

$$R_0F.e=f_1^2$$

and since $FR=f_1$, we obtain

$$\frac{KR}{\rho}=\frac{\frac{f_1^2}{e}+f_1}{\rho-\rho_0}$$

Also $JR_0=\rho_0$ is the image of $L_1S=\rho$, hence

$$\frac{\rho}{\rho_0}=\frac{e}{f_1}\ \ldots\quad\ldots\quad\ldots\quad(1)$$

and

$$KR=\frac{f_1(\rho+\rho_0)}{\rho-\rho_0}\quad\ldots\quad\ldots\quad(2)$$

from which the extremity K of the critical distance is determined.

Further

$$RR_0=KR\frac{\rho-\rho_0}{\rho}$$

$$=\frac{f_1(\rho+\rho_0)}{\rho}\quad\ldots\quad\ldots\quad(3)$$

For an emmetropic eye to look through the instrument without accommodation, the light must emerge from the second lens as a parallel beam; that is, the light from the object point P must be made by the first lens to unite in a point P' which is the focal point of the second lens; thus

$$P'S=f_2$$

Ex. 88.

The lens at the object end of a tube of diameter 2 cm. and length 10 cm. has a focal length of 4 cm. Where is the critical distance, and what is the size of ρ_0?

$$f_1=4\text{ cm.};\ \rho=1\text{ cm.};\ e=6\text{ cm.}$$

hence

$$\rho_0 = \frac{\rho f_1}{e} = \frac{1 \times 4}{6} = \frac{2}{3} \text{ cm.}$$

from (2)

$$KR = \frac{4\left(1 + \frac{2}{3}\right)}{1 - \frac{2}{3}} = 20 \text{ cm.}$$

Thus the point K lies relatively far from the instrument.

Ex. 89.

With the same instrument as in Ex. 88 the object lies 24 cm. from the pole of the objective. Where are the entrance and exit pupils and what is the focal length f_2 of the eye-piece?

The critical distance extends to a point 20 cm. from the objective, so that in this case the object point P lies outside it, and GJ is consequently the entrance pupil of the instrument. From equation (3)

$$RR_0 = 4\left(1 + \frac{2}{3}\right) = 6\tfrac{2}{3} \text{ cm.}$$

The image P' of P formed by the objective is given by the fundamental equation for thin lenses;

$$\frac{1}{RP'} = \frac{1}{f_1} - \frac{1}{RP} = \frac{1}{4} - \frac{1}{24} = \frac{5}{24}$$

or $RP' = 4 \cdot 8$ cm.

Hence the focal length f_2 of the eye-piece is

$$10 - 4 \cdot 8 = 5 \cdot 2 \text{ cm.}$$

The exit pupil coincides with the eye-piece aperture of the tube.

Ex. 90.

As in Ex. 89, except that the object distance $PR = 12$ cm.

In this case the object point lies within the critical distance, so that the objective aperture of the tube is also the entrance pupil. To find the position of P' we have

$$\frac{1}{RP'} = \frac{1}{f_1} - \frac{1}{RP} = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}; \quad \therefore RP' = 6 \text{ cm.}$$

Hence $f_2 = 10 - 6 = 4$ cm.

To find the exit pupil, we must find the image of the objective aperture formed by the eye-piece. Representing the centre of the exit pupil by R',

$$\frac{1}{SR'} = \frac{1}{f_2} - \frac{1}{SR} = \frac{1}{4} - \frac{1}{10} = \frac{3}{20}; \quad \therefore SR' = 6\tfrac{2}{3} \text{ cm.}$$

To find the radius ρ' of the exit pupil,

$$\frac{\rho'}{\rho} = \frac{SR'}{SR} = \frac{20}{3 \times 10} = \frac{2}{3}; \quad \therefore \rho' = \tfrac{2}{3} \text{ cm.}$$

Ex. 91.

As in Ex. 89 only $PR = 2$ cm.

The object point again lies outside the critical distance, so that the entrance pupil is GJ, the image of the eye-piece aperture formed by the objective.

$$\frac{1}{RP'}=\frac{1}{4}-\frac{1}{2}=-\frac{1}{4}\,;\ RP'=-4\text{ cm.}$$

The image point P' is thus virtual and lies 4 cm. to the left of R. The focal length of the eye-piece must therefore be $f_2=10+4=$ 14 cm. in order that an emmetropic unaccommodated eye sees a sharp image.

Ex. 92.

As in Ex. 89 except that $PR=6$ cm

In this case P lies within the critical distance so that the objective aperture is the entrance pupil of the instrument. To find the position of P'

$$\frac{1}{RP'}=\frac{1}{4}-\frac{1}{6}=\frac{1}{12}\,;\ RP'=12\text{ cm.}$$

Thus the image P' lies outside the tube, hence the focal length of the eye-piece will be $f_2=-2$ cm.; that is, the eye-piece is negative.

To find the exit pupil we must find the image of the objective aperture LL_2 formed by this negative lens.

$$\frac{1}{SR'}=\frac{1}{f_2}-\frac{1}{SR}=-\frac{1}{2}-\frac{1}{10}=-\frac{3}{5}\,;\ SR'=-1\tfrac{2}{3}\text{ cm.}$$

Thus the exit pupil lies within the tube, and is not directly accessible to the eye.

69. Fundamental Equations for Optical Instruments.

All the essential properties of an instrument such as illumination, field of view, resolving power, etc., depend on the positions of the entrance pupil and exit pupil. Hence in the following we will measure distances from the centres of the pupils after we have given a few examples to show how the positions of these centres are determined.

Fig. 58 represents an optical instrument. R and R' are the centres of the entrance pupil and exit pupil of radii ρ

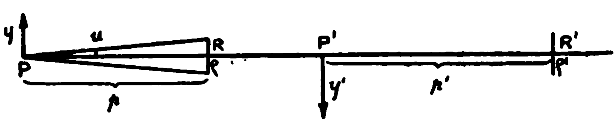

FIG. 58.

and ρ'. At P, at a distance $PR=p$ from R, is an object y. An image y' at the point P' is formed by the whole instrument, the distance of P' from R' being $R'P'=p'$.

To an eye looking into the instrument the image y' will appear sharp when it is at a distance from the eye corresponding to the latter's refraction condition. The conditions in the figure are for a myopic or accommodated eye.

From § 40 we have

$$\frac{n\rho^2}{p}+\frac{n'\rho'^2}{p'}=D\rho\rho \quad \ldots \quad \ldots \quad (4)$$

and

$$\frac{y'}{y}=\frac{np'}{n'p}\cdot\frac{\rho}{\rho'} \ldots \quad \ldots \quad \ldots \quad (5)$$

where n and n' are the refractive indices in the object and image spaces respectively, and D is the power of the instrument.

The eye is usually in air, so that $n'=1$, and these equations become:

$$\frac{n\rho^2}{p}+\frac{\rho'^2}{p'}=D\rho\rho' \quad \ldots \quad \ldots \quad (6)$$

$$\frac{y'}{y}=\frac{np'}{p}\cdot\frac{\rho}{\rho'} \ldots \quad \ldots \quad \ldots \quad (7)$$

These two equations are the fundamental equations for the theory of optical instruments.

70. Magnification of Optical Instruments.

Avoiding, in the definition of magnification, the use of angles, we may define the magnification as the quotient of the sizes of two retinal images; namely, that image produced by the instrument, and the image produced when the eye is brought to a certain conventional distance d from the object (the instrument being removed). Simple as this definition appears, its exact mathematical statement presents certain difficulties. The latter image will be taken as the "normal image" and, as we shall see below, for a given object has a fixed magnitude for all eyes, whilst the retinal image formed by the instrument depends on the refraction and accommodation conditions of the eye as well as on the optical system of the instrument.

We will now assume that all eyes have the same first focal length f and the same second focal length f'; that is, only axial ametropia is exhibited. A glass placed at the first focal point of an ametropic eye may reduce the ametropia to zero and aids the eye, so that an object y at the distance d may be seen sharply without accommodation.

Thus, if in equation (11*a*) § 58, we put $x=o$ which is the case considered here, the magnitude K is equal to unity; that is, there is no change in the size of the retinal image produced by the glass.

$$\text{Retinal Image} = \frac{y\,f}{d}.$$

Thus we see that **the size of the retinal image is independent of the refraction condition of the eye, and, when d is fixed, gives the required normal image so long as the eye is able to form this retinal image.**

This definition removes all difficulties which up to the present had to be overcome in the definition of a "conventional projection distance," and is extremely clear in that the person using an optical instrument can make quantitatively clear in every case what magnification he obtains, and in most cases is even able to prove it experimentally.

Gullstrand's expression for the magnification is $\frac{u_k}{y}$, where y is the size of object and u_k is the angle subtended at the first principal point of the eye by the image produced by the instrument. From equation (15) §60, $y'=u_k.\frac{a'}{n}=\frac{u_k}{A'}$, we may write this expression

$$\frac{u_k}{y}=\frac{y'}{y}.A',$$

where y' is the size of the retinal image of the object y. The quotient $\frac{y'}{y}$ represents the lateral magnification and thus the required magnification is equal to the product of the lateral magnification and the principal point vergence in the image space. The advantage of Gullstrand's definition lies essentially in the fact that A' changes only very little during accommodation, so that the magnification in the case of vision with accommodated eyes is given very approximately by the above equation.

Distances will here be referred to the centres of the entrance pupil and exit pupil as being by far the most important points of an optical instrument.

Suppose now that the same object is viewed through the instrument, and for the sake of generality assume that the eye is not placed exactly at the exit pupil of the instrument, but that the first focal point F of the eye is at a distance e from R' (Fig. 58); then we obtain an image of size y' at a distance $p'+e$ from the first focal point of the eye. If the latter be accommodated, its focal length will be, say f_0. In this way an image is produced on the retina whose size is given by

$$\text{Retinal Image with Aided Eye} = \frac{y' f_0}{p'+e}.$$

It is the quotient of these two retinal images which we call the magnification v; thus

$$v = \frac{y' f_0}{p' + e} \cdot \frac{d}{yf}$$

$$= \frac{y'}{y} \cdot \frac{f_0}{f} \cdot \frac{d}{p' + e}.$$

Transposing from equation (7)

$$v = \frac{np'}{p} \cdot \frac{\rho}{\rho'} \cdot \frac{f_0}{f} \cdot \frac{d}{p' + e} \quad \dots \quad \dots \quad \dots \quad (8).$$

Had the distances been referred to the principal points of the instrument instead of the centres of the pupils, equation (8) would have been simplified considerably. Excluding accommodation, $f = f_0$; and calling e the distance from the focal point of the eye to the second principal plane of the instrument

$$v = \frac{y' f}{a' + e} \div \frac{yf}{d} = \frac{y'}{y} \cdot \frac{d}{a' + e}$$

or

$$v = \frac{na'}{a} \cdot \frac{d}{a' + e}$$

and since

$$\frac{n}{a} - \frac{1}{a'} = D \text{ and } \frac{1}{a' + e} = L$$

where D is the power of the instrument and L is the spectacle refraction of the ametropic eye, we obtain

$$v = d\left[D + L\left(1 - e\,D\right)\right] \dots \quad \dots \quad \dots \quad (8a)$$

which holds for all optical instruments except the telescope.

71. Classification of Optical Instruments for Visual purposes.

In the widest sense every optical system which brings about an alteration in the path of a bundle of rays is an optical instrument.

In a restricted sense, various well-defined types can be distinguished according to the purpose to which they are to be put :—

1. Telescopes: instruments which yield magnified retinal images of distant objects.
2. Magnifying glasses and microscopes, for the purpose of magnifying near and small objects.

In addition to these two well-developed types the necessities of recent times have brought into being in turn further instruments, in which the magnification is not of primary importance, or in which the objects are neither very far distant nor very near. Amongst these are found, on the one hand, the sighting arrangements for naval and other artillery, and those instruments used for obtaining the largest possible view of the horizon ; on the other hand, instruments for the observation of the interior of bodies, such as the cystoscope, and the later forms of the ophthalmoscope, &c.

72. Magnification and Aperture for an Unaccommodated Emmetropic Eye.

An optical instrument for visual purposes when produced commercially, is so adjusted that an emmetropic eye at rest receives sharp retinal images ; *i.e.* the images in the instrument must appear very distant. This requirement corresponds to the condition

$$p' = \infty$$

Since there is no accommodation

$$f = f_0$$

Taking into consideration these two assumptions, we obtain from equations (6) and (8)

$$\frac{n\rho}{p} = D\rho' \qquad \dots \qquad \dots \qquad (9)$$

and

$$v = \frac{n\rho}{\rho'} \cdot \frac{d}{p} = Dd. \qquad \dots \qquad \dots \qquad (10)$$

Thus, in general, the magnification of an optical instrument of finite power D is, for an unaccommodated emmetropic eye, equal to the power of the system multiplied by the conventional distance of vision, and is independent of the distance of the eye from the exit pupil of the instrument.

If u be the angle subtended by the radius ρ of the entrance pupil at the object point P, then for small angles

$$u = \frac{\rho}{p} \quad \dots \qquad \dots \qquad \dots \qquad (11).$$

In general, the product of this angle u and the refractive index in the object space is called the Numerical Aperture a of an instrument.

$$\textit{Numerical Aperture } a = nu$$
$$= \frac{n\rho}{p}$$

Hence

$$a = D\rho' \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

Thus, in general, in an optical instrument the numerical aperture is equal to the product of the power of the system and the radius of the exit pupil.

73. Illumination of Optical Instruments.

What is meant by "illumination" is somewhat complex and difficult to define. In the first place the brightness of a surface emitting light is, in general, different in different directions; it depends on the quality and colour of the surface. Moreover, subjective influences such as the sensitiveness to light of the retina, the duration of the light stimulus, &c., play an important part.

In optical instruments, however, we are usually concerned with the paths of those rays in the neighbourhood of the optic axis, so that we can ignore the different intensities in different directions. Further, since we do not wish to compare the illumination with various eyes, nor the various physiological effects in the eye, we can obtain, *with reference to optical instruments*, a comparatively simple and clear idea of "illumination."

In vision with the unaided eye, more light enters the eye from every point of the object space the greater is the pupil, or more correctly, the greater is the entrance pupil of the eye. If the radius of the entrance pupil be ρ_a, its area is $\pi\rho_a^2$. Thus if the radius be doubled or trebled, then the illumination is increased four or nine times. Since in converging the axes of vision the human pupils contract, the eye sees near objects less bright than distant ones. In the darkness the pupils expand, by which means the eye transforms itself, as the occasion demands, into an instrument of higher illumination.

The value of ρ_a may vary from one millimetre to four mm. In normal daylight we may assume $\rho_a = 1{\cdot}5$ to 2 mm.

In Fig. 59, LL_1 is a screen in which a small circular opening of diameter $\alpha\beta = 2\rho'$ has been pricked with a needle; if the entrance pupil of the eye, whose diameter is $AB = 2\rho_a$, be brought close to this opening, the eye looking through the opening experiences a considerable diminution of light due to the fact that the cross section of the incident bundle is smaller than the entrance pupil of the eye, and the latter

is no longer filled with rays coming from individual points in space. Evidently the illumination H_0, when vision is unhindered, is greater than the illumination H in the case of vision through the small opening, in the ratio of the areas $\pi\rho_a^2$ and $\pi\rho'^2$ of the entrance pupil and incident bundle respectively. Hence the relation

$$H = \frac{\rho'^2}{\rho_a^2} . H_0 \quad \dots \quad (13).$$

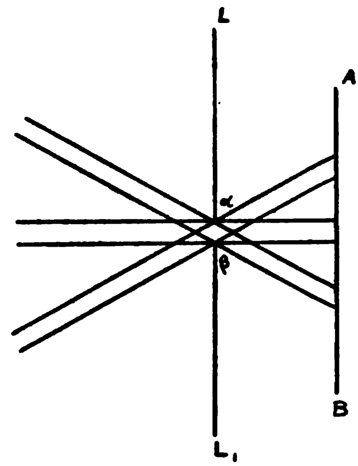

FIG. 59.

This equation is true for all optical instruments, the radii of whose exit pupils equal ρ', so long as the latter is smaller than the radius ρ_a of the entrance pupil of the eye. If the exit pupil of the instrument be equal to the entrance pupil of the eye, then $\rho' = \rho_a$ and $H = H_0$, that is, the illumination with the instrument is the same as in unaided vision. If the exit pupil of the instrument be greater than the entrance pupil of the eye, there is no increase of illumination since of each of these bundles of greater cross section, only such portion enters the eye as corresponds to the entrance pupil of the eye.

The loss of light due to absorption within the instrument has here been neglected.

Ex. 93.

What is the illumination and magnification of the instrument in Ex. 90?

The radius of the *exit pupil* of this instrument is $\rho' = \frac{2}{3}$ cm. $= 6\frac{2}{3}$ mm.

so that ρ' is usually greater than ρ_a, and $H = H_0$. Thus the illumination is the same as in vision with the unaided eye. The instrument consists of two lenses each of 4 cm. focal length, that is of power 25 *Dp.*, separated by a distance of 10 cm. $= 0 \cdot 1$ m.; hence the power of the whole system is given by

$$D = 25 + 25 - 0 \cdot 1 \times 25^2 = 50 - 62 \cdot 5 = -12 \cdot 5.$$

If we neglect the sign, which merely shows whether the image is erect or inverted, we have from equation (10), putting $d = 25$ cm. or $\frac{1}{4}$ m,

$$v = 12 \cdot 5 \times \frac{1}{4} = 3 \cdot 1$$

Thus the instrument magnifies $3 \cdot 1$ times.

For the intensity of illumination of the image on the retina, *see* the Author's "Einführung in die medizinische Optik," Leipzig, 1904; and Archiv. fur Optik, Pt. 1, p. 211. In the first it is also proved that a luminous surface appears equally bright at all distances. Also this brightness of a surface can never be increased by using an optical instrument. In the case of point objects such as the fixed stars, however, their brightness increases proportionately with the objective aperture of the observing telescope.

References.

R. S. Heath ... "Geometrical Optics," Camb. Univ. Press. 2nd Ed., 1895, Chap. X.

Southall ... "Principles and Methods of Geometrical Optics." Macmillan. 1913. Chap. XIV.

P. Drude ... "Theory of Optics," translated by C. R. Mann and R. A. Millikan. Longmans Green. 1907.

CHAPTER VIII.

The Magnifying Glass.

74. Magnifying Glass in combination with an Emmetropic Unaccommodated Eye.

The magnifying glass is generally a simple or spherically and chromatically corrected convex lens, which is usually not mounted in a tube.

From equation (10) § 72 the magnification for an emmetropic eye at rest is given by

$$v = Dd$$

Since the conventional distance d is usually taken as 25 cm. = 0·25 m.

$$v = \frac{D}{4}.$$

The magnification* of a magnifying glass is equal to one quarter of the power expressed in dioptres.

The object lies at the first focal plane and the cones of rays from the object points are converted into cylinders of rays in the image space.

Ex. 94.

What is the magnification of a convex lens of focal length 12 cm.?

$$D = \frac{100}{12}. \quad \text{Hence } v = \frac{100}{48} = 2\tfrac{1}{12}.$$

Ex. 95.

A meniscus has radii 10 and 20 cm. What is the magnification of the lens, if $n = 1{\cdot}5$?

$$\text{The reciprocal of the focal length} = \frac{1}{2}\left(\frac{1}{10} - \frac{1}{20}\right) = \frac{1}{40}\text{cm.}$$

$$\text{Hence } D = \frac{100}{40} = \frac{5}{2}\,Dp. \text{ and consequently } v = \frac{5}{8}.$$

75. The Verant.

In order to view with the unaccommodated eye and in natural sizes, photographs which have been taken with a

* Called the "absolute" magnification $v = \frac{D}{4}$.—Trans.

comparatively small objective focal length (say 10–15 cm.), we make use of the magnifying glass. To find the magnitude of the focal length of such a magnifying glass, it is necessary to observe the following considerations :

If the eye is fixed on a particular distant object in space, which appears under the visual angle u, an image is formed on the retina of size

$$y' = u\,f$$

where f is the first focal distance of the eye.

If we photograph the object with a lens of focal length ϕ, a photograph of size $u\phi$ is formed. If the latter be observed in parallel rays (that is with no accommodation) through a magnifying glass of focal length ψ, this image of size $u\phi$ in the focal plane of the magnifying glass is seen under the visual angle u_1 where

$$u_1 = \frac{u\phi}{\psi}$$

and the size of the retinal image y'' is given by

$$y'' = \frac{u\phi}{\psi} \cdot f$$

In order that the image obtained when using the magnifying glass may be the same size as the one obtained with the unaided eye, then $y' = y''$, and ϕ must be equal to ψ.

The focal length of the magnifying glass must then be equal to the focal length of the objective with which the photograph is taken.

For the eye at rest the path of the rays is fixed by the entrance pupil of the eye, whose image, produced by the magnifying glass, is to be taken as the entrance pupil of the whole system, magnifying glass plus eye.

Since, however, in observing an image, the eye makes rolling movements round its *point of rotation*, which lies about 13 mm. behind the vertex of the cornea, this point may be considered as a centre of perspective, and it is advantageous to move the point of intersection of the principal rays in the image space to this point for the fulfilment of the orthoscopic condition ; (*cf.* § 48 Gullstrand's Condition).

Apparatus of this kind have been designed by M. v. Rohr of the firm of Zeiss, Jena, and are known as **Verants.**

Commercially, the lenses of the verant are made with focal lengths 11 and 15 cm. In order to fix the position

of the eye so that the point of rotation of the latter takes up the proper position, special eye shields are provided.

The binocular magnifying glass made by Zeiss* consists of a double telescope with a convex lens in front of it (Fig. 60). It permits of the observation of objects by a decreased convergence of the visual axes (*see also* Chapter X).

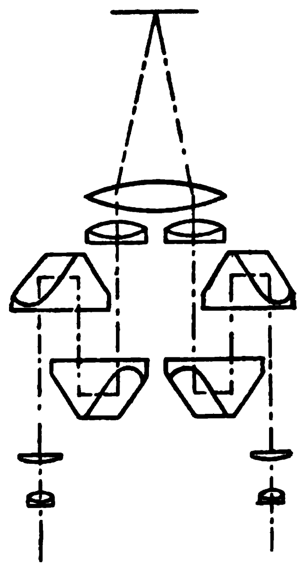

FIG. 60.

76. Magnifying Glass in Combination with an Ametropic Eye.

When the magnifying glass is considered alone, that is independently of the system of the eye, the entrance pupil in most cases is formed by the lens holder, and since this lies usually very close to the first principal point, the exit pupil and entrance pupil are very nearly equal in size, so that we may put $\frac{\rho'}{\rho} = 1$. Then in equation (8) § 70 if we put $\frac{f_0}{f} = 1$, which is the case for unaccommodated vision with an axial ametropic eye, and also putting $n = 1$,

$$v = \frac{p'}{p} \cdot \frac{d}{p' + e}.$$

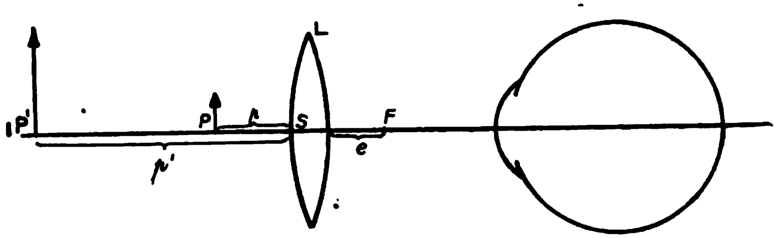

FIG. 61.

In Fig. 61, let L be a magnifying glass, assumed to be a thin lens with vertex S, and in which the exit pupil and

* German Patent 392047.

entrance pupil coincide. P is the object point, P' the image point, then

$$PS=p \qquad P'S=p' \qquad SF=e,$$

where e is the distance of the vertex S from the first focal point F of the eye.

Since $n=1$ and p' is to be reckoned negative, equation (6) § 69 gives

$$\frac{1}{p}-\frac{1}{p'}=D.$$

Consequently we obtain the "individual magnification"

$$v=\frac{(Dp'+1)\,d}{p'+e} \quad \ldots \quad \ldots \quad (1)$$

The quantity $\frac{1}{p'+e}$ represents here the focal point refraction L of the eye. Equation (1) may now be written in the form, since $\frac{1}{p'+e}=L$:

$$v=[D+L(1-eD)]d.$$

If now $e=0$, the vertex of the lens coincides with the first focal point of the eye, and representing the magnification by v_0*, and putting $d=\frac{1}{4}$ m. then

$$v_0=\frac{D+L}{4} \quad \ldots \quad \ldots \quad (2)$$

The individual magnification of a magnifying glass placed in the normal position is equal to the sum of the power of the glass and the focal point refraction of the eye, divided by four.

Ex. 96.

If the eye be myopic 10 *Dp.*, what will be the magnification produced with a magnifying glass of 20 *Dp.*?

$v_0=\frac{20+10}{4}=7\frac{1}{2}$ whilst the absolute magnification is $\frac{D}{4}=5$.

Ex. 97.

What is the magnification for a hypermetrope of 10 *Dp.* provided with a magnifying glass of 20 *Dp.*?

$$v_0=\frac{20-10}{4}=2\cdot5.$$

* Called the "individual normal magnification," v_0.—Trans.

If the eye does not take up the normal position relative to the glass, the magnification depends on the value of e and must be derived from equation (1). Substituting the value for v_0 from equation (2) in equation (1) we have

$$v = v_0 \cdot \frac{p'}{p' + e} \quad \ldots \quad \ldots \quad (3)$$

For myopia, p' is positive; when e is positive, then the distance of the glass from the eye is greater than the normal distance. In this case $\frac{p'}{p'+e} < 1$ and the magnification is less than the individual normal magnification. When e is negative, the glass assumes a position between the eye and the first focal point and we obtain an increase in the magnification.

For hypermetropia, p' is negative and the above relations are reversed.

Ex. 98.

A glass of power 20 *Dp.* is placed 5 cm. from the first focal point of a myopic eye of refraction $L = 10$ *Dp.* What is the magnification?

$$p' + e = \frac{1}{L} \text{ if } p' \text{ and } e \text{ be expressed in metres.}$$

Then

$$p' = 0{\cdot}1 - 0{\cdot}05 = 0{\cdot}05$$

and

$$v = v_0 \frac{0{\cdot}05}{0{\cdot}05 + 0{\cdot}05} = 0{\cdot}5 v_0.$$

CHAPTER IX.

The Telescope.

77. General Properties.

The simple optical instrument shown in Fig. 57 becomes a telescope if the object point lie at infinity. The point P', the image of P, becomes the focal point of the objective in consequence; but since P' must also be the focal point of the eye-piece for an emmetropic eye at rest to receive sharp images, it is characteristic of this type of instrument that

the focal planes of objective and eye-piece must coincide.

The telescope thus belongs to the class of centred systems which we have already described as telescopic. (§ 33). To obtain the magnification of the telescope, we must compare the image of the distant object formed by the unaided eye, with the image formed by using the telescope. We may thus put the conventional distance d, at which we view an object with the unaided eye, equal to p. The difference between d and p, that is the distance from the entrance pupil of the instrument to the eye, becomes negligible in comparison as these distances increase.

If in equation (10), Chapter VII, we put $d = p$, then when $n = 1$

$$v = \frac{\rho}{\rho'}, \quad \dots \quad \dots \quad \dots \quad (1)$$

which is a quite general relation.

The expression $\frac{\rho}{\rho'}$ represents the magnification at conjugate points, which from equation (27b) § 24 is constant and equal to the quotient of the focal lengths of the two sub-systems; in this case, the objective and eye-piece. Since the focal lengths of the objective and eye-piece are f_1 and f_2 respectively, we have

$$v = \frac{\rho}{\rho'} = \frac{f_1}{f_2} \quad \dots \quad \dots \quad (2)$$

78. Cases in which the Objective Aperture is not the Entrance Pupil.

Generally the objective aperture of a telescope is the entrance pupil. Cases, however, arise in which a special

investigation on the lines of § 67 is necessary. As an example, we may choose a simple instrument as represented in § 68. In this case $e = f_2$ and equation (1) Chapter VII may be written

$$\frac{\rho_0}{\rho} = \frac{f_1}{f_2} = v$$

or

$$\rho_0 = v\rho \quad \dots \quad \dots \quad \dots \quad (3)$$

Equation (2) § 68 viz.

$$K\,R = \frac{f_1\,(\rho + \rho_0)}{\rho - \rho_0}$$

now becomes

$$K\,R = f_1 \frac{\left(1 + \frac{\rho_0}{\rho}\right)}{\left(1 - \frac{\rho_0}{\rho}\right)} = f_1 \left(\frac{1 + v}{1 - v}\right) \quad \dots \quad (4)$$

If v be greater than unity ($v > 1$), then from equation (3) $\rho_o > \rho$, or *the objective aperture is also the entrance pupil of the instrument.*

In this case $K\,R$ becomes negative, that is K lies to the right from R (Fig. 57). The critical distance extends outwards from F to infinity towards the left, and then from infinity on the right to the point K. The object consequently *always* lies within the critical distance.

If v be equal to unity ($v = 1$) equation (3) becomes $\rho_o = \rho$. The secondary entrance pupil $G\,J$ in Fig. 57 has the same size as the objective aperture; there are then two entrance pupils.

If v be less than unity ($v < 1$), then $\rho_o < \rho$ and the secondary aperture, GJ in Fig. 57, becomes the entrance pupil of the instrument, and is conjugate to the eye-piece aperture $L_1\,L_3$. The value of KR becomes finite and positive.

An infinitely distant object point lies no longer within the critical distance. We have an example of this latter case when looking through a telescope in the reversed direction.

79. Telescope in combination with an Ametropic Unaccommodated Eye.

If the eye be ametropic, p' is finite; the objective and eye-piece must be moved further apart, or brought closer together, p' receiving a finite value. Since the focal planes

of the eye-piece and objective no longer coincide, but have been displaced by an amount Δ (= optical tube length), the telescope has a finite focal length and consequently a power D differing from zero.

$$D = \frac{\Delta}{f_1 f_2}$$

MAG = $\frac{\text{TUBE LENGTH}}{F_1 \times F_2}$ $9 = \frac{18}{f_1 f_2}$

where f_1 and f_2 are the focal lengths of the objective and eye-piece. In equation (8) § 70, put $d = p$ for the case of a distant object. The quotient $\frac{f_0}{f} = 1$, and since $n = 1$ we obtain

$9 f_1 \ldots$

$F_1 f_2 = 2$

$F_1 = 12''$

$$v = p' \cdot \frac{\rho}{\rho'} \cdot \frac{1}{p' + e} \quad \ldots \quad \ldots \quad (5)$$

$f_2 = \frac{2}{12} = \frac{1}{6}''$

From equation (6) § 69, if $p = \infty$

$$\frac{\rho}{\rho'} = \frac{1}{p'D} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Hence equation (5) may be written

$$v = \frac{1}{D} \cdot \frac{1}{p' + e} \quad \ldots \quad \ldots \quad (7)$$

but $L = \frac{1}{p' + e}$ is the spectacle refraction, therefore

$$v = \frac{L}{D} = \frac{f_1 f_2 L}{\Delta} \quad \ldots \quad \ldots \quad (8)$$

Hence :

For sharp focus the magnification of a telescope is equal to the quotient of the spectacle refraction and the power of the telescope.

The second form of equation (8) gives the magnification in terms of the change of length Δ and the quantities f_1, f_2 and L.

A simplified formula is obtained by assuming that $\frac{\rho}{\rho'} = v_\infty$,* an assumption which is not rigorously true, since ρ' changes slightly in consequence of the small displacement of the eye-piece; we thus obtain the approximate relation

$$v = v_\infty \cdot \frac{p'}{p' + e} = v_\infty (1 - eL) \quad \ldots \quad (9).$$

---

* **That is the magnification when the instrument is used in conjunction with an emmetropic eye.—Trans.**

In the case of telescopes possessing a real exit pupil, the latter being made to coincide with the entrance pupil of the eye, e becomes negative; and since the entrance pupil of the eye is distant 3 mm. from the vertex of the cornea, that is $15{\cdot}5 + 3 = 18{\cdot}5$ mm. from the first focal point of the eye, we find that $e = -0{\cdot}02$ m., from which

$$v = v_\infty (1 + 0{\cdot}02\ L) \quad \ldots \quad \ldots \quad (10).$$

To a myopic eye the magnification therefore appears to be increased; and decreased to hypermetropes.

Ex. 99.

What is the change in the absolute magnification of a telescope if a myope of 10 *Dp.* spectacle refraction observes an object?

Equation (10) gives

$$v = 1{\cdot}2\ v_\infty$$

80. Astronomical Telescope.

The astronomical telescope in its simplest form consists of two positive lenses which are so placed that the second focal plane of the first coincides with the first focal plane of the second. In Fig. 62 these lenses are assumed very thin for the sake of simplicity. Hence the principal planes of these two lenses, represented by the lines L_1 and L_2 perpendicular to the axis, coincide at the respective vertices B and G. A beam of parallel rays from a distant object enters the first lens of aperture AC at an inclination w. The principal ray passes through the thin lens at the vertex B and is consequently not deviated. The incident parallel beam unites in the common focal plane at E, where the inverted image ED is formed. These three rays now diverge from E, and meet the eye lens (the eye-piece) in the three points $J\,K\,S$, and since E lies in the focal plane of the latter lens, they must emerge as a parallel beam. In order to find the direction of emergence of this beam, we must consider a ray from E passing through the vertex G—its direction is fixed, and further, it is not deviated in passing through the eye lens. The bundle of parallel rays emerging from the points $J\,K\,S$, must be parallel to this latter direction, since in the paraxial region all rays from E must travel, after refraction through the eye lens, parallel to one another. The emergent principal ray from K cuts the axis in N, and since this ray passed through the axial point B in the object space, B and N are conjugate points. Now since the principal rays of all incident bundles in the object space at all inclinations pass through the point B, they

must, after refraction through the telescope, pass through the point N. This point N is called the **Eye point.** Now

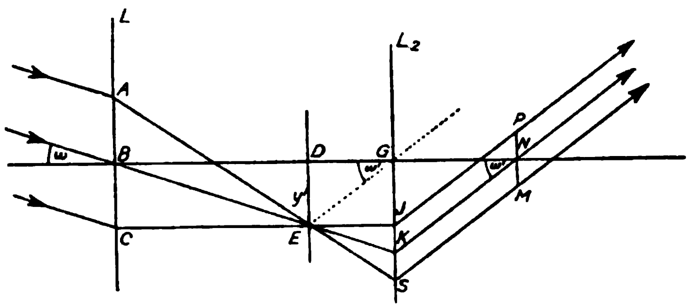

FIG. 62.

every principal ray passing through N is the centre of a cylindrical bundle of rays all of which must intersect in a circle of diameter PM, perpendicular to the axis. This circle appears as a luminous disc to an eye placed at a certain distance from the telescope; it is called the **Ramsden Circle.** The fundamental property of the telescope is shown by Fig. 62, which has been constructed from quite general considerations. The incident parallel bundle makes an angle w with the axis, and finally emerges at an angle $w' = G\hat{N}K = D\hat{G}E$. If the focal length of the objective is f_1, and of the eye-piece f_2, then it is easily seen that

$$w = \frac{y'}{f_1} \qquad w' = \frac{y'}{f_2}$$

where y is the size of the image ED. Hence

$$\frac{w'}{w} = v = \frac{f_1}{f_2}$$

where v is the magnification of the telescope. From similar triangles, $\frac{AC}{PM} = \frac{f_1}{f_2}$, so that

$$v = \frac{AC}{PM},$$

or, the magnification* is equal to the diameter of the objective divided by the diameter of the Ramsden Circle; that is, the quotient of the entrance pupil and exit pupil of the telescope. The position of the exit pupil is easily deduced

* This is the principle involved in the use of the "dynameter," an instrument for measuring magnification.—Trans.

by finding the image of B formed by the eye lens; the longer the focal length of the latter, the further the point N recedes from the eye lens, and the greater must be the aperture of the same in order not to decrease the apparent field of view.

81. The Field-Lens. Eye-pieces of Ramsden and Huygens.

In most cases a second lens at a certain distance from the objective is added to the eye lens, whose function is to control the path of the principal rays. Such a lens is called a **Field-Lens.** To illustrate its action, consider such a positive lens placed at D (Fig. 62) in the common focal plane of the objective and eye lens. Since the principal plane of this field-lens very nearly coincides with the plane of the real image y', there results no appreciable change in the magnification of the image, and therefore the emerging rays from E must enter the image space again as a parallel beam, and further, at the same inclination w with the axis. Each principal ray, in consequence of the refraction at the field-lens, suffers a change of direction; the rays no longer intersect the axis in the point N, but in a point which is nearer to the eye lens. The result is that the effective aperture of the eye lens is smaller, but there is no loss of field. Further, magnification and illumination, which latter depends only on the diameter of the exit pupil, remain unchanged. If the field-lens be situated outside the common focal plane, there results a combined effect in which the paths of the rays forming the image, including the principal rays themselves, are modified. The **Ramsden eye-piece** belongs to the type in which the field-lens lies in the common focal plane and has the same focal length as the eye lens. The **Huygens eye-piece** is the most widely used of the second type—the field-lens lies outside the common focal plane. Both eye-pieces consist of plano-convex lenses, and, owing to the choice of focal length, they are very nearly achromatic. In the Huygens eye-piece the curved surfaces of the lenses face the objective, whilst in the Ramsden they face each other. The following data apply to these eye-pieces.*

* The Ramsden eye-piece given in this Table is not what is usually termed a Ramsden eye-piece in this country. From a reference to Parkinson's "Optics," or Edser's "Light," it will be seen that the constructional data of a Ramsden are: focal lengths of field and eye lenses respectively, f; distance between the lenses $\frac{2}{3}f$. The Huygens also is different in this country, the data being: focal length of field-lens : focal length of eye lens : separation :: 3 : 1 : 2.—Trans.

	Huygens.	Ramsden.
Focal length of complete eye-piece	f	f
Focal length of field lens ...	$\frac{3}{2}f$	f
Focal length of eye lens	$\frac{3}{4}f$	f
Distance between the two lenses...	$\frac{9}{8}f$	f
Distance of the eye from last lens	$\frac{1}{4}f$	o
Distance of stop from field lens ...	$\frac{1}{2}f$	o

In the case of the Ramsden eye-piece the field-lens is moved a little out of the focal plane in order to move the eye point to a required position.*

82. Kellner Eye-piece.

In the Ramsden eye-piece, besides the chromatic error, there is a disturbing effect due to distortion. The **Kellner** eye-piece removes both errors by cementing together in the eye lens, a plano-concave lens of greater relative dispersion and high refractive index to a bi-convex lens of small relative dispersion and low refractive index. The converging lens of the eye-lens system faces the field-lens of the eye-piece so that the cemented concave surface is directed towards the latter (Fig. 63). There exists, however, the disadvantage of this eye-piece, that it lacks a certain sharpness of image. The Zeiss eye-piece† removes this last defect, by combining two kinds of glass in the eye-lens, so that as compared with the Kellner eye-piece, there is a great dissimilarity between their relative dispersions, whilst the difference between the refractive indices is very small. Suitable glasses are the ordinary silicate crown (for the first converging lens), and a flint of lower refractive index (for the diverging lens). Barium silicate crown and

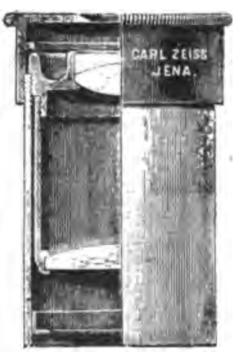

FIG. 63.

* Thus any dust on the field lens will not be in focus.—Trans.
† German Patent No. 179473.

ordinary silicate flint are also suitable. The constructional data for such an eye-piece as represented in Fig. 64 are :—*

Radii, Thicknesses, and Distances.

$r_1 = \pm\infty$	$l = 26{\cdot}3$
$r_2 = -88{\cdot}10$	$d_I = 22{\cdot}0$
$r_3 = +73{\cdot}42$	$m = 78{\cdot}3$
$r_4 = -49{\cdot}92$	$d_{II} = 23{\cdot}0$
$r_5 = \pm\infty$	$d_{III} = 4{\cdot}9$

Glass.

	I	*II*	*III*
$n_C =$	1·50801	1·58846	1·60810
$n_F =$	1·51605	1·59816	1·62470

In a modified form of the Ramsden eye-piece, the eye-lens system consists of a single converging lens and of a double lens which may be either diverging or converging. The power of this double lens is not more than half the power of the single lens, and the concave side of its cemented surface faces the single lens.†

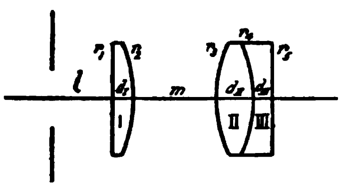

FIG. 64.

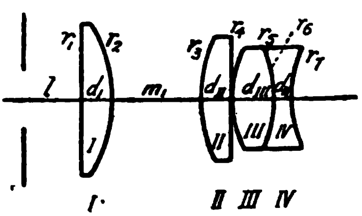

FIG. 65.

The constructional data for such a lens as shown in Fig. 65 are given as follows :—‡

Radii, Thicknesses, and Distances.

$r_1 = \pm\infty$	$l = 34{\cdot}8$
$r_2 = -100$	$d_I = 15{\cdot}0$
$r_3 = +80{\cdot}3$	$m_1 = 80$
$r_4 = \pm\infty$	$d_{II} = 15{\cdot}5$
$r_5 = +125$	$m_2 = 0$
$r_6 = -76$	$d_{III} = 17{\cdot}5$
$r_7 = +200$	$d_{IV} = 5{\cdot}0$

* Another construction of the Kellner eye-piece is described in the Brit. Patent No. 24009, 1903.

† German Patent No. 188200. C. Zeiss, Jena.

‡ Further modifications of the Ramsden eye-piece are found in the German Patents Nos. 184614 and 184615.

Glass.

	I, III	*II*	*IV*
$n_C=$	1·50762	1·58703	1·60814
$n_F=$	1·51559	1·59673	1·62474

83. Other Forms of the Astronomical Eye-piece.

Figs. 66 and 67 represent two more forms of the astronomical eye-piece called respectively the monocentric and the

FIG. 66.

FIG. 67.

orthoscopic. The first is of the same type as the Steinheil magnifying glass, whilst the second is a modified form of the Ramsden eye-piece.

In order to observe the sun through a telescope, the intensity of the light must be diminished. A practical form of the astronomical eye-piece designed by Colzi and constructed by Zeiss for this purpose, is represented in Fig. 68, and (in section) in Fig. 69. The sunlight traverses first of all a long focal length objective to the negative lens E, and is partially reflected by the plane surface of transparent glass S. The mirror Q deflects the light and heat rays which penetrate S. The light which is reflected from S passes on to the double prism P, consisting of a right angled glass prism B and a liquid prism C. In consequence of the small difference of refractive index of the prisms, only a small amount of the light is reflected at the surface of separation, whilst the greater part passes

through the transparent membrane *D*. *X* represents the field diaphragm which lies at the first focal plane of the eye-piece.

FIG. 68.

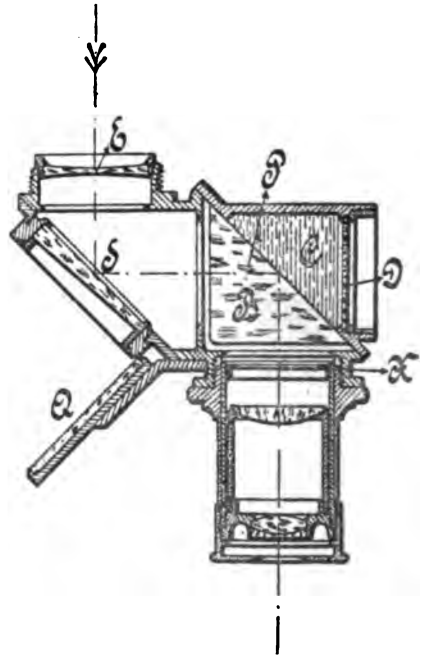

FIG. 69.

84. The Terrestrial Eye-piece.

The astronomical eye-piece furnishes an inverted* image. In order to erect the image, the terrestrial eye-piece, consisting of four plano-convex lenses (Fig. 70) as developed by Fraunhofer, served for a long time.

FIG. 70.

* It must be remembered that besides a simple inversion top to bottom, there also results an inversion left to right, *i.e.*, reversion.—Trans.

Terrestrial eye-pieces are comparatively long, and further, are longer, the longer is their focal length. In the Fraunhofer construction their length amounts to about 10 times the focal length. Although terrestrial eye-pieces of shorter construction, with similar optical performance, have recently been developed, especially at the works of Merz, Steinheil, and Voigtlænder, yet for portable telescopes prismatic inverting systems are used. In Fig. 70, the two plano-convex lenses on the left form the inverting system, whilst the two on the right represent essentially a Huygens eye-piece. (*See also* the paragraph on Sighting Telescopes.)

85. Prismatic Telescopes. General Remarks.

As is well known, a plane mirror causes a reversion of the image of an object (right to left and vice versâ). If we subject the reverted image to another reflection at a second plane mirror so placed as to invert the image (top to bottom), we obtain the effect of the inverting system of a terrestrial eye-piece. For technical reasons the surfaces of prisms, and particularly of totally reflecting prisms, are used as reflecting elements. Modern telescope construction, and the improvements in modern rangefinders, have given rise to a great number of prism combinations in whose design the inversion and reversion of the images, as well as considerations of the economy of space, have been solved.

On the condition that the incident axial ray, after passing through the instrument, shall emerge parallel to itself, there is a whole series of solutions, the most important of which we will now consider.

86. Porro-prisms.

Of Porro-prism systems (so called after their discoverer, Porro, a French engineer) there are two types.

The **Porro-system of the first type** is shown in Fig. 71. It consists usually of two isosceles right angled prisms whose hypotenuse faces are rectangles with sides in the ratio of 1 : 2. The prisms are cemented together crosswise on their hypotenuse faces. The incident ray a_1 is totally reflected, first at the two faces I and II of the first prism, and then at the faces III and IV of the second prism, emerging parallel to its original direction as the ray a_4.

Placed between the objective and eye-piece of an astronomical telescope, this system produces the same effect as an inverting system of lenses.

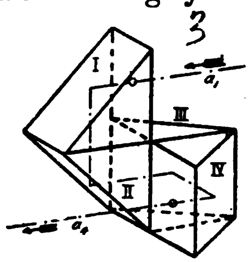

FIG. 71.

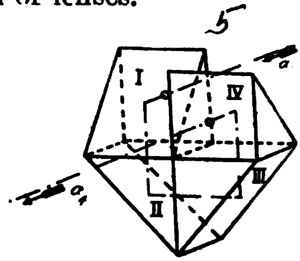

FIG. 72.

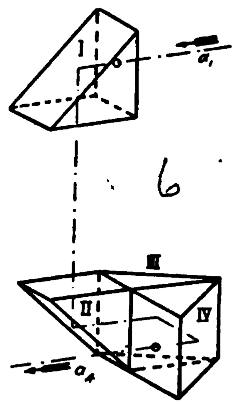

FIG. 73.

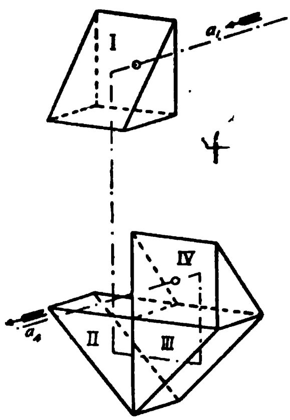

FIG. 74.

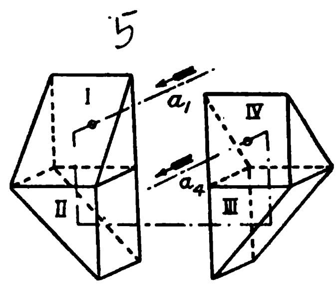

FIG. 75.

The **Porro-system of the second type** is represented in Figs. 72 and 75. It consists of two congruent tetrahedra (sphenoids) placed in opposition, shown cemented in Fig. 72 and separated in Fig. 75. The incident ray a_1 is totally reflected at faces 1 and 2 of the first sphenoid, and then at the faces 3 and 4 of the second sphenoid, finally emerging parallel to its original direction as the ray a_4. The action is precisely the same as in the Porro-system of the first type.

From these two types of Porro-systems, it is possible to devise new forms for special purposes. Figs. 73 and 74 represent modifications of both types, in which right angled prisms are separated from the Porro-system. The reflecting faces I and II may be separated to any extent in the direction of the normal to the separating face. It is clear that these arrangements may be employed to place the objective in a higher plane than the eye-piece. This is employed in telescopes for observing over a protective

embankment. In order to obtain a sideways displacement of the objective and eye-piece, two separated sphenoids of the second type of Porro-system, as shown in Fig. 75, have been employed.*

87. Optical Prisms in General.

Modern optical developments have produced a whole series of glass elements with reflecting surfaces. In most cases total reflection is employed. At the same time silvered surfaces are also used, depending on the various purposes to which the elements are to be applied. Their function may be to revert or invert the optical image, to alter the paths of the rays, etc. Elements of this kind are simply called optical **prisms** and a number of the more important elements are described below. In Figs. 76 to 86, the principal ray is drawn through the prism, and the small circles denote the points of incidence and emergence of this ray. Frequently two consecutive reflections are obtained in a prism, the reflecting faces of which include an angle, usually 90°. Such a prism, on account of its resemblance to a roof, is known as a **Roof Prism.**

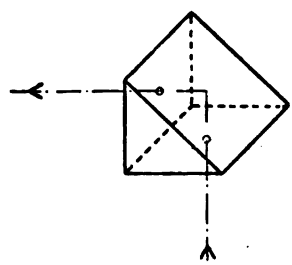

FIG. 76.

88. Erecting Prisms. Pentagonal. Amici Prism.

Fig. 76 shows a simple totally reflecting right-angled prism; the reflection takes place at the hypotenusal face, the principal ray striking the incident and emergent faces

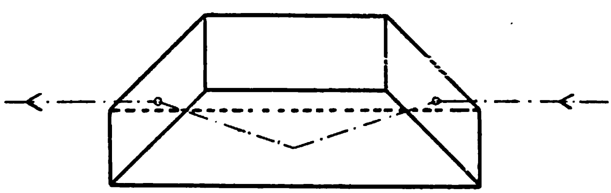

FIG. 77.

normally. The **Erecting rhomboid prism** (Fig. 77) produces an inversion of the image when viewed through directly, the incident ray suffering refraction at the inclined incident and emergent faces. The **Pentagonal** prism (also called the Prandl

* *See* a paper by S. Czapski, 1895, "Ueber neue Arten von Fernrohren, insbesondere zum Handgebrauch."

prism) shown in Fig. 78, deviates the ray without producing either inversion or reversion of the image.* The Amici prism

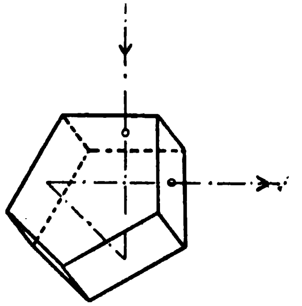

FIG. 78.

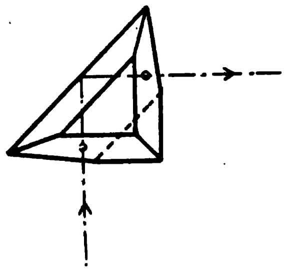

FIG. 79.

(Fig. 79) is essentially a tetrahedron. The principal ray passes normally through the first surface, is then reflected at a roof surface of 90° and emerges perpendicular to the last surface. The path of the ray is thus deviated by 90°. On account of these reflections, there results a complete inversion of the image, and its action is thus comparable with that of a convex lens when the latter forms an image of a distant object.

89. Prisms producing complete Inversion of Image without bending the path of the Ray.

(Abbe, Sprenger, Daubresse, [Hensoldt], Dove, Wirth.)

We are here concerned with prismatic elements which produce complete image inversion, whilst the incident and emergent rays are parallel or even collinear.

Fig. 80 shows the Abbe prism. It will be seen that the incident and emergent principal rays lie in the same straight

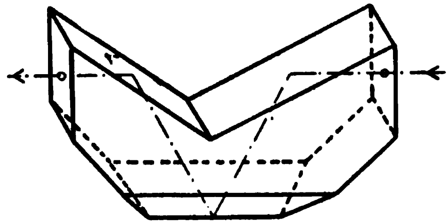

FIG. 80.

* The pentagonal prism is extensively employed in most types of Rangefinders (*see* Chapter XI). It can easily be demonstrated that the supplement of the angle between the incident and emergent rays is always twice the angle between the reflecting faces measured in the plane of the rays. This prism is sometimes wrongly called a Prandl prism ; so far as is known, its practical introduction is due to the French artillerist Colonel Goulier.—Trans

line after suffering the various reflections, including two at the roof surface, shown underneath. In the Sprenger prism (Fig. 81) the roof surface is seen to the left: it will further be observed that in this case, as also in the cases of Figs. 82 to 84, the principal ray is displaced laterally and emerges

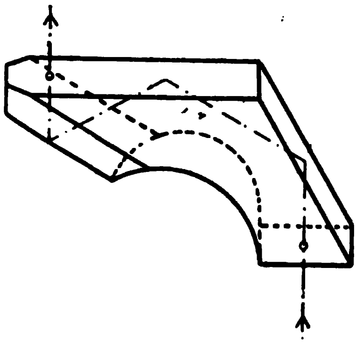

FIG. 81.

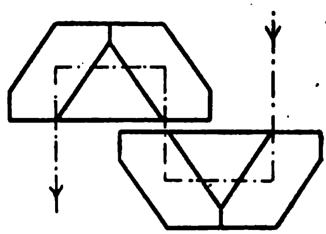

FIG. 82.

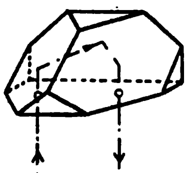

FIG. 83.

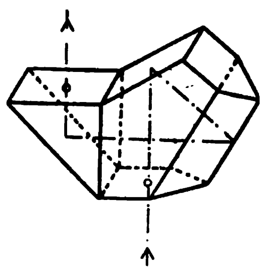

FIG. 84.

parallel to its original direction. The prisms of Daubresse, Fig. 82, act in exactly the same way as a Porro-prism system of the first type, which they also resemble in shape. Fig. 83 shows the actual path of the ray. These prisms of Daubresse possess a further important property, namely, that the prisms may be rotated in any way relative to each other without affecting the optical performance, provided the two adjacent surfaces remain parallel to one another.*

Fig. 84 represents the Hensoldt-(Huet-) prism, which may be considered as consisting of a pentagonal, one of whose reflecting surfaces (the right in the figure) is ground to a roof edge, whilst on the last transmission face is cemented a right angled prism.

90. Prisms of Wirth and Dove.

For the sake of completeness we show in Figs. 85 and 86 further types of prisms which up to the present, however,

* German Patent, No. 104149.

have not been used in telescope construction. Fig. 86 shows a direct vision Amici prism giving a complete erection of the image and may be employed as a Porro-prism. It has been called a direct-vision Amici prism,* but it is better described after its inventor as the Wirth prism.

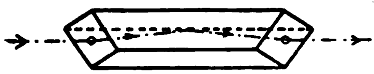

FIG. 85.

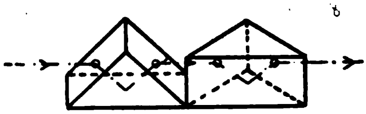

FIG. 86.

Prisms arranged as in Fig. 86 also give complete erection of the image as with a Porro-system. This combination has been known up to the present as the Dove prism. M. von Rohr has given it the title "Harting-Dove Reversion Prism."†

91. Construction of Prismatic Telescopes.

Until re-discovered by Abbe, the prismatic telescopes of Porro and others had been forgotten. The re-discovery of this important feature in telescope construction is due to Abbe, and to the firm of Carl Zeiss, Jena. Instruments of this kind have been constructed by other German firms, including Goertz, Voigtlaender, Busch, Hensoldt, Schütz, Leitz, &c., and also by firms of other countries. Zeiss, Goerz, Voigtlaender and Busch, make use, at least for the ordinary field glass, of the Porro-system of the first type. These firms do not vary essentially in their mechanical arrangements of the parts. The firm of Zeiss favours a separate focussing of each eye-piece, whilst other firms arrange that both eye-pieces are focussed simultaneously.‡

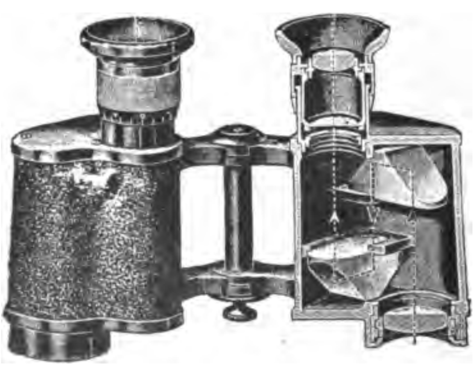

FIG. 87.

* German Patent 85971. † "Die binokularen Instrumente," p. 61.
‡ As well as separately.—Trans.

Fig. 87 represents glasses made by Zeiss, and Fig. 88 one made by Goerz.*

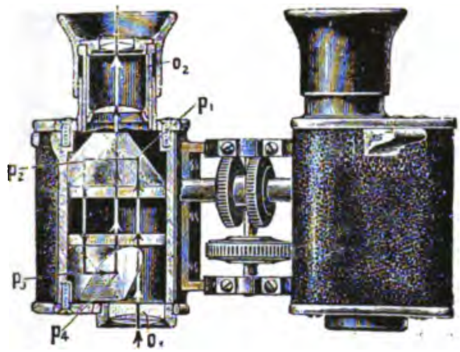

FIG. 88.

The three power eye-piece telescope of the form shown in Fig. 89 is a Zeiss production, the image-erecting prisms

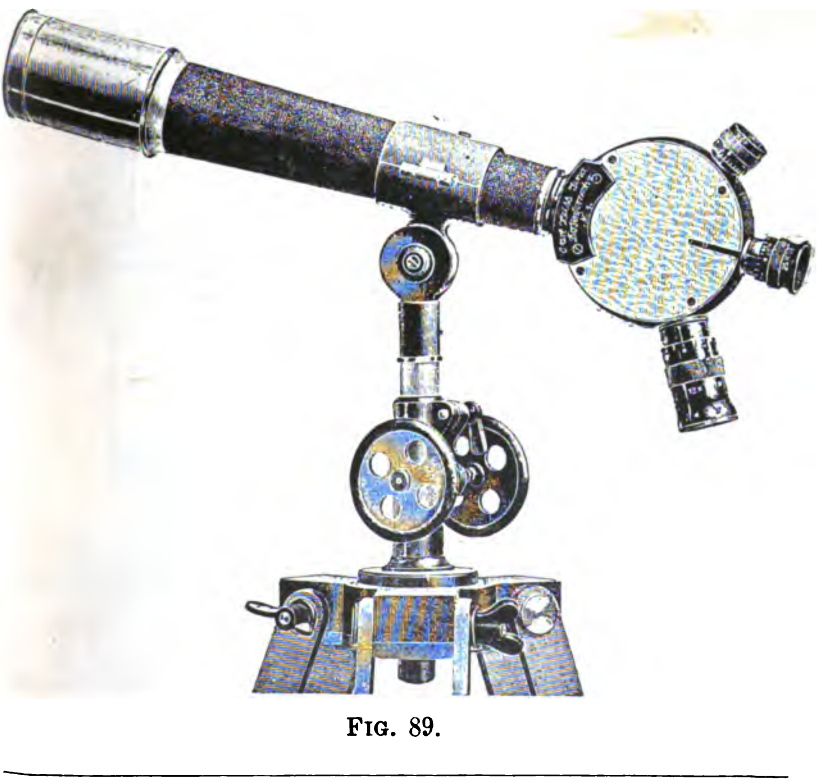

FIG. 89.

* Fig. 88 shows the original form of the Goerz glasses which are now made of the standard pattern with central hinge.—Trans.

being enclosed in a box-like compartment. There are three eye-pieces which by rotation may be placed as required in the path of the rays in order to give different magnifications. This kind of telescope is used by the Army and Navy for observation purposes in particular. Since the axial ray emerges from the eye-piece parallel to its original direction, the directing (sighting) of the telescope is thereby facilitated. The erection of the image is obtained by means of the Abbe prism (*see* § 89).

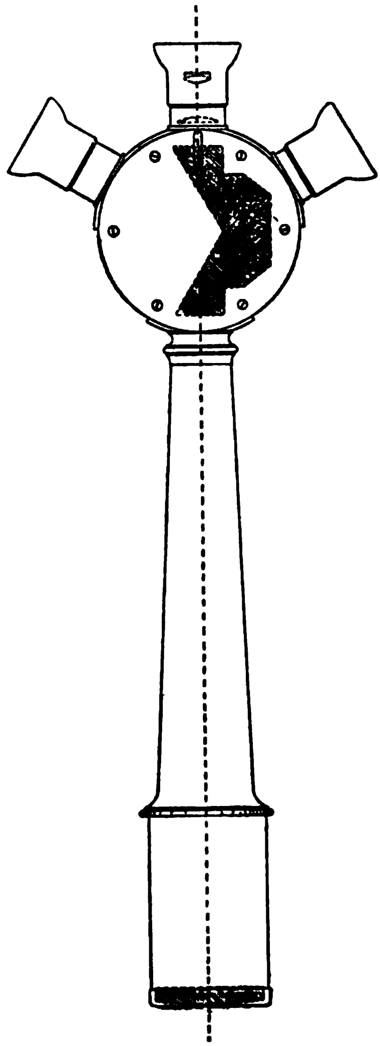

FIG. 90.

The path of the rays in such a telescope is shown in Fig. 90.

Other types of telescopes by Zeiss possess the property of being able to vary the stereoscopic effect or "Relief."* As will be seen later (§ 105), the stereoscopic effect may be increased in the case of double telescopes, when the distance between the objectives is increased. The Zeiss Relief Telescope is constructed with this in view; it is shown in Fig. 91 with its arms extended, and in Fig. 92 with its arms closed. In the latter case it may be used for observing over embankments.

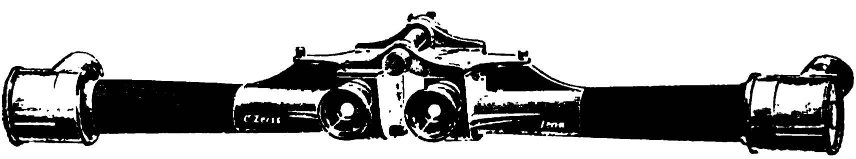

FIG. 91.

* *See* Chap. X. on Stereoscopy.—Trans.

Fig. 93 shows the path of the rays through such a telescope.

FIG. 92.

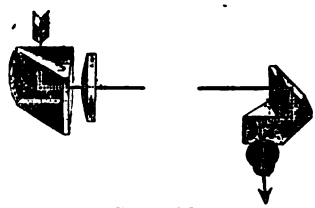

FIG. 93.

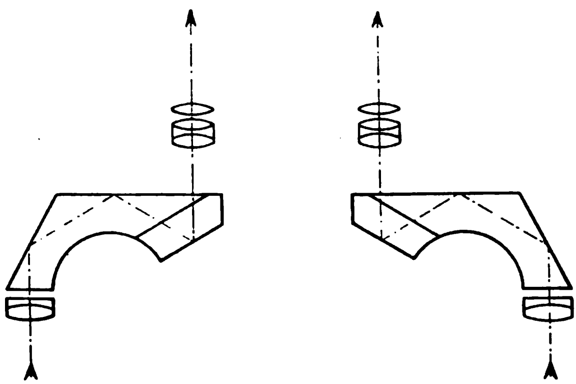

FIG. 94.

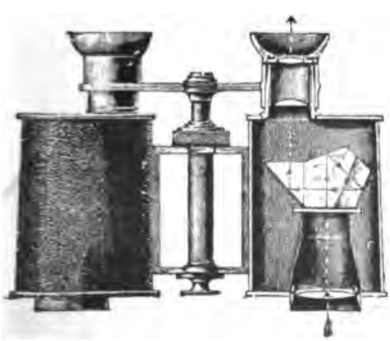

FIG. 95.

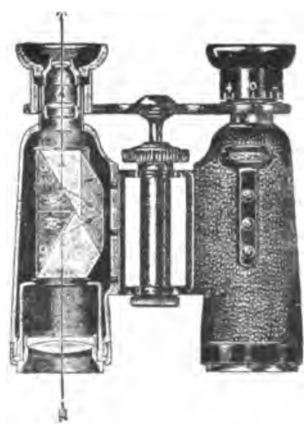

FIG. 96.

Field glasses in which the stereoscopic effect is increased by increasing the objective distance, as is usually the case for prism glasses, have been introduced by Zeiss under the trade name of "Teleplaste." In these glasses each telescope contains a Sprenger prism for the purpose of erecting the image (*see* § 89).

The path of the rays in the "Teleplaste" is shown in Fig. 94.

Fig. 95 represents the optical arrangement in the prismatic field glass of Hensoldt, Wetzlar. The Hensoldt prism employed has already been described (§ 89).

In another type of field glass by the same firm (Fig. 96) a prism is used which corresponds essentially with that first devised by Abbe.*

In the prismatic telescope of Schütz of Cassel, a prism is used which corresponds essentially to the second type of Porro-system.

In Fig. 97 the path of the rays is shown in the left hand tube, whilst the right hand telescope is shown dismantled.

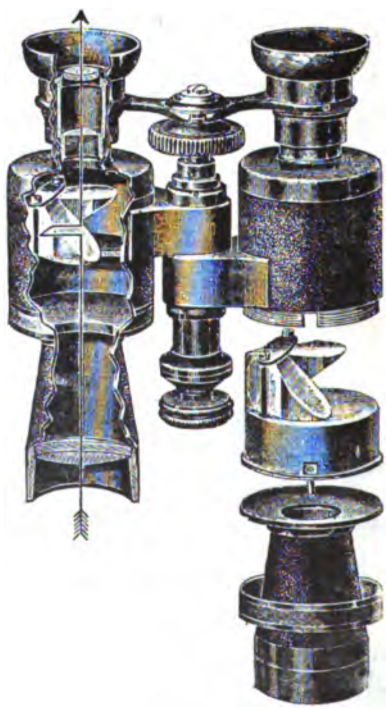

FIG. 97.

* (*See* Czapski "Ueber neue Arten von Fernrohren für den Handgebrauch." Gesellschaft für Mechanik und Optik. 1895. No. 10.)

92. Short Prismatic Telescope by Busch.

In the shortened telescope of Busch, the light is reflected at two right-angled isosceles prisms, which are arranged symmetrically to the axis of the telescope with their hypotenuse faces parallel. As shown in Figs. 98 and 99, the telescope consists of an objective a, and an eye-piece

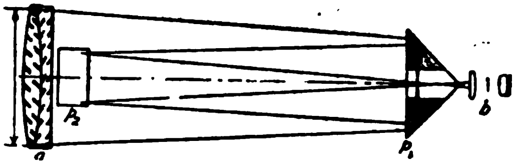

FIG. 98.

b, as in the case of the astronomical telescope. The two isosceles prisms p_1 and p_2 are placed in the path of the rays; the former, p_1, has a hole bored through it whose diameter is equal to the diameter of the eye-piece, in order to allow

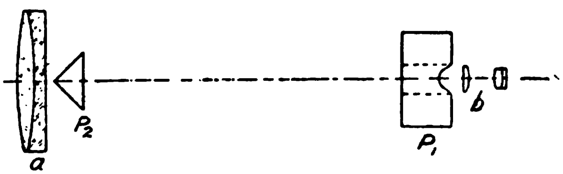

FIG. 99.

the rays to pass through it. Figs. 98 and 99 show two sections of the telescope.

93. The Dutch Telescope. (Galilean.)

The Galilean telescope consists of a converging system (objective) and a diverging system (eye-piece) whose focal planes coincide. The image formed is erect (Fig. 100).

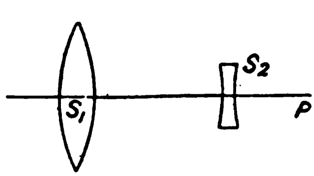

FIG. 100.

Assuming that the objective and eye-piece are thin lenses, with vertices at S_1 and S_2, whilst F represents the common focus, the length of the instrument is equal to

$$l = f_1 - f_2.$$

As in the case of all telescopes, the magnification is given by the quotient of the focal lengths of the objective and eye-piece, or, by the quotient of the radii of the entrance pupil and exit pupil.

The positions of the entrance and exit pupils are not found so simply as heretofore, depending largely on the point of view adopted for their consideration. If the Galilean telescope be considered alone without reference to the eye, we may take the objective aperture as the entrance pupil, whilst the position of the exit pupil coincides with that of the image produced by the eye-piece of the vertex S_1. Let x denote the distance of the exit pupil from the vertex S_2, then

$$\frac{1}{l}+\frac{1}{x}=-\frac{1}{f_2}$$

from which it follows,

$$x=-\frac{lf_2}{l+f_2}=\frac{-f_2\,(f_1-f_2)}{f_1}=-\frac{l}{v}.$$

The negative sign means that the exit pupil lies to the left of S_2 and is therefore *virtual*. Thus the entrance pupil of the eye, or the point of rotation of the latter, cannot be made to coincide with the exit pupil of the instrument, as is the case with the astronomical and terrestrial telescopes. Consequently the field of view is reduced, and the eye must be displaced to receive the required portion of the emergent beam.

A better conception of the course of the rays is obtained by considering the eye in conjunction with the telescope.

In Fig. 101, L_1 is the objective and L_2 the negative eye-piece, which, for simplicity, are represented by straight lines. $A'R'B'$ represents the entrance pupil of the eye with centre R'. The image of $A'R'B'$ to the left through the whole system is *virtual* and lies at $A\,R\,B$. This latter acts as the entrance pupil of the system. The incident bundle of rays filling this entrance pupil and represented by 1, 2, and 3, passes through the exit pupil of the system as the rays 1′, 2′, and 3′.

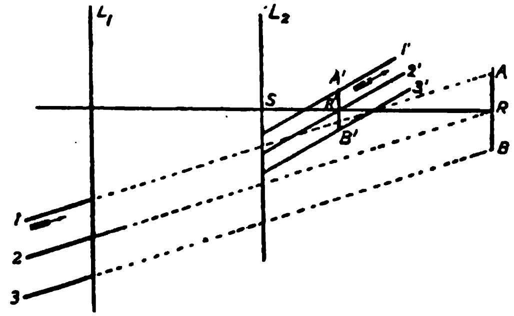

FIG. 101.

It will be seen in this arrangement that the bundles effective in forming the image pass through quite different parts of the objective and eye-piece, and that R' is the effective point of intersection of the principal rays. In order to correct, particularly for astigmatism and chromatism, for the portion of the beam more remote from the axis, an average value of 3 cm. for $S R'$ is chosen, where S is the vertex of the eye-piece, corresponding to the position of the point of rotation of the eye.

94. Illumination of the Telescope.

From equation (13) § 73, the illumination H through the telescope is given by

$$H = \frac{\rho'^2}{\rho_a^2} . H_0 \quad \ldots \quad \ldots \quad (11).$$

Where

H_0 is the natural illumination (*i.e.*, with the unaided eye),

ρ' is the radius of the exit pupil of the instrument, and

ρ_a is the radius of the entrance pupil of the eye.

The illumination H thus increases as ρ' increases (assuming that ρ_a and H_0 are constant) until ρ' becomes equal to ρ_a. Neglecting losses of light by reflection and absorption, the illumination through the telescope is then the same as with the unaided eye. In this case the pupil of the eye is completely filled with rays, and further increase of ρ' serves no purpose. In darkness the pupil dilates, and thus ρ_a is a variable quantity. In ordinary daylight ρ_a has a value of 1·5 mm. ; consequently all telescopes with exit pupil of diameter 3 mm. or over, have normal illumination. The pupil of the eye in twilight has a diameter of 7 mm. and therefore only those telescopes whose exit pupils are 7 mm. in diameter possess normal brightness when used during twilight. Modern prismatic telescopes, especially those required to be easily carried by hand, are constructed so that the exit pupil is 3–5 mm. in diameter, whilst Galilean telescopes (opera glasses) usually have an exit pupil of 7 mm. and over. From this point of view, the latter are termed "night glasses."

Ex. 100 .

At night the radius ρ_a of the pupil of the eye is 4 mm. A prismatic telescope which magnifies five times and in which $\rho' = 1·5$ mm. is compared with a Galilean telescope in which $\rho' = 5$ mm. and magnification of $\times$ 5. What is the ratio of their illuminations and what are the sizes of their objectives ?

The effective exit pupil of the Galilean telescope has a radius of 4 mm. From equation (11) the illumination in the two telescopes under the existing circumstances are proportional to the squares of the radii of the effective exit pupil, *i.e.* $1{\cdot}5^2 : 4^2$ or $1 : 7{\cdot}1$. Hence the illumination of the Galilean telescope is 7·1 times that of the prismatic telescope. The objective aperture 2ρ of the prismatic telescope amounts to 15 mm. from the relation $\rho = \rho' v$. For the Galilean it is 50 mm.

95. Field of View of the Telescope.

In the case of a simple astronomical telescope consisting of two lenses, the boundary of the field of view is in general due (*see* § 80) to the rim of the eye lens. When an astronomical or terrestrial eye-piece is employed, the limitation of the field of view is effected by a diaphragm placed in the eye-piece. Here it is presumed that the entrance pupil of the eye coincides with the exit pupil of the instrument. If this be not the case, there is a diminution of the field of view, which becomes more pronounced the further the eye is removed from the exit pupil of the instrument. In the Galilean telescope the exit pupil of the instrument may be taken as lying between the objective and eye-piece (§ 93), and consequently when the magnification is increased somewhat over 4, the field of view rapidly diminishes.*

In order to give the Galilean telescope a greater field of view the objective may be composed of several lenses at considerable distances from each other; two lenses have been employed in which both have been made positive, or the first lens positive and the second negative. The gain in the field, however, is only slight.†

96. Dutch Telescopes of Zeiss (Galilean).

The field of view of a Galilean telescope is directly dependent on the *aperture ratio* (*i.e.*, the quotient of the aperture and the focal length) of the objective. With an increasing aperture ratio somewhat above 1/3, the corrections for chromatism, astigmatism, and distortion become more difficult. It must be remembered that the removal of these errors is to be effected only for those principal rays which intersect at the point of rotation of the eye, *i.e.*, R' (Fig. 101). Telescopes of this kind with a considerable field of view,

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* For the calculation of the field of view in the Galilean telescope *see* the Author's "Lehrbuch der geometrischen Optik," 1902, p. 280.

† For further particulars concerning the calculation of the field for such a combined objective, *see* the Author's "Lehrbuch der geometrischen Optik," Leipzig, 1902, p. 283, *et seq.*

and corrected for the above defects, have recently been patented by Carl Zeiss*. In the following table, the con-

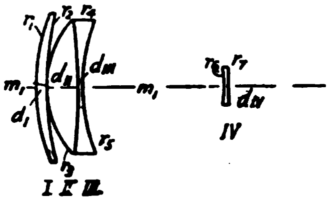

FIG. 102.

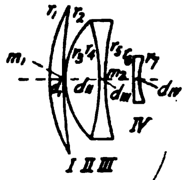

FIG. 103.

structional data for two such instruments, shown in Figs. 102 and 103, are given.

I. Glass.

$n_{DI}=n_{DII}=n_{DIV}=1{\cdot}5726$	$\nu_I=\nu_{II}=\nu_{IV}=57{\cdot}5$
$n_{DIII}=1{\cdot}6245$	$\nu_{III}=35{\cdot}8$

Radii, Thicknesses, and Distances.

$r_1=+\ 90{\cdot}0$	$d_I=5{\cdot}0$
$r_2=+185{\cdot}0$	$m_1=0{\cdot}0$
$r_3=+\ 43{\cdot}0$	$d_{II}=13{\cdot}2$
$r_4=-\ 92{\cdot}0$	$d_{III}=1{\cdot}0$
$r_5=+\ 91{\cdot}0$	$m_2=57{\cdot}0$
$r_6=-\ 28{\cdot}1$	$d_{IV}=1{\cdot}0$
$r_7=+\ 28{\cdot}1$	

II. Glass.

$n_{DI}=1{\cdot}5163$	$\nu_I=6{\cdot}40$
$n_{DII}=1{\cdot}6099$	$\nu_{II}=58{\cdot}9$
$n_{DIII}=1{\cdot}6103$	$\nu_{III}=37{\cdot}2$
$n_{DIV}=1{\cdot}5825$	$\nu_{IV}=46{\cdot}4$

Radii, Thicknesses, and Distances.

$r_1=+\ 48{\cdot}0$	$d_I=5{\cdot}0$
$r_2=+156{\cdot}5$	$m_1=0{\cdot}0$
$r_3=+\ 25{\cdot}0$	$d_{II}=11{\cdot}0$
$r_4=-120$	$d_{III}=1{\cdot}0$
$r_5=+\ 44{\cdot}7$	$m_2=9{\cdot}8$
$r_6=+\ 66{\cdot}5$	$d_{IV}=1{\cdot}0$
$r_7=+\ 11{\cdot}5$	

* Brit. Patent No. 2340, 1910.

97. Sighting Telescopes.

Simultaneously with the development of the explosive and steel industries, the means for accurate sighting of artillery and rifles have made extraordinary strides. The time is not far distant when a sighting telescope will become a necessary accessory for artillery.* Sighting telescopes have assumed widely differing forms according to the particular purpose in view. These forms conform partly to the old terrestrial telescope, and partly to the prismatic. In certain cases, as for example when taking aim in torpedo boats and submarines, telescopes may have a length of several metres, whilst as gun sights and for use in armoured turrets, they may be very compact.† Further, the form of the telescope may be modified, as for example, for observing over a parapet. In the case of sighting telescopes for small arms, the weight of the instrument is also an important consideration. Ordinary sighting with the unaided eye necessitates a rapid change of accommodation on the part of the marksman between the open and back sights. Consequently the eye becomes strained, and the steadiness in taking aim suffers. In the sighting telescope, an image of the distant object is formed in the focal plane of the eye-piece, where there is also a cross-wire. The necessity for accommodation thus disappears. Moreover the accuracy in taking aim is increased by the telescope magnification.

The usual type tends towards the old terrestrial telescope. Care must be taken, however, that the exit pupil of the instrument lies comparatively far away from the eye-piece so that the eye will not be damaged by any recoil.

In Fig. 104 let L_1 be the objective, L_2 the inverting

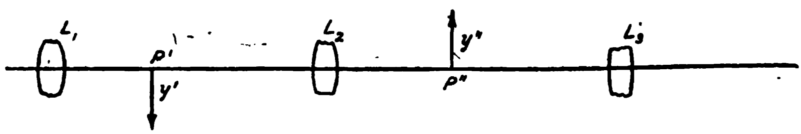

FIG. 104.

system, and L_3 the eye-piece. For the sake of a better correction of the image, L_2 and L_3 often consist of two equal lenses placed rather close together.

The objective must be corrected for spherical and chromatic aberration, whilst the remaining lenses are usually achromatic. The lenses L_1, L_2, and L_3 have focal lengths f_1, f_2, and f_3 respectively.

* Most guns are now provided with optical sights.—Trans.

† Possibly Dr. Gleichen refers to periscopes for the direction of submarines.—Trans.

A distant object which subtends the visual angle ω at the eye, forms an image P' whose size $y' = \omega f_1$. The lateral magnification of the inverting system is $\beta = \frac{y''}{y'}$ so that the size of the image P'' is

$$y'' = y'\beta = \omega f_1 \beta.$$

The image y'' lies in the focal plane of the eye-piece L_3, and appears through the same under the angle ω'; therefore

$$y'' = \omega' f_3$$

But

$$\frac{\omega'}{\omega} = v$$

where v is the telescope magnification.

Hence

$$v = \frac{f_1}{f_3}\beta \quad \ldots \quad \ldots \quad (1)$$

If the distances of the points P' and P'' from the principal points of the lens L_2 be a and a', then

$$\beta = \frac{a'}{a} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where

$$\frac{1}{a} + \frac{1}{a'} = \frac{1}{f_2} \ldots \quad \ldots \quad \ldots \quad (3)$$

Provided that the aperture of the inverting system be not too small, the objective aperture is the entrance pupil of the system and may be assumed to lie at the first principal plane of the objective. In order to find the position of the exit pupil of the instrument, we must first find the image of the entrance pupil formed by the inverting system L_2. Since the distance of the objective from L_2 is $(f_1 + a)$, then

$$\frac{1}{x_1} = \frac{1}{f_2} - \frac{1}{f_1 + a} \quad \ldots \quad \ldots \quad (4)$$

where x_1 is the distance of the required image from L_2. This image now becomes the object for the eye-piece L_3, its distance from the eye-piece being $a' + f_3 - x_1$. Hence if x denote the distance of the exit pupil from the second principal point of the eye-piece, we have

$$\frac{1}{x} = \frac{1}{f_3} - \frac{1}{a' + f_5 - x_1} \quad \ldots \quad \ldots \quad (5)$$

3

By means of equations (1) to (5) the magnification and the position of the exit pupil of the instrument may be determined.

If the length, magnification, and position of the exit pupil of the telescope have been previously decided on, these equations enable us to determine the focal lengths of the lenses required. As in the case of all telescopes, the diameter of the exit pupil is equal to $\frac{1}{v}$ times the objective aperture. Generally the eye-piece is quite filled with rays, and hence the subjective or *apparent* field of view may be obtained by setting out straight lines from the centre of the exit pupil to the extremities of a diameter of the eye-piece rim. The vth part of this angle gives the objective or *true* field of view. Since the centre of the exit pupil is comparatively distant from the eye-piece, the latter must have a correspondingly large aperture in order to provide a convenient size of field.

We will now consider the case when $\beta = 1$, *i.e.*, when the inverting system L_2 produces no magnification.

It follows from (1)

$$v = \frac{f_1}{f_3} \quad \dots \quad \dots \quad \dots \quad (6)$$

and from (2) and (3)

$$a = a' = 2f_2$$

After eliminating x_1 from equations (4) and (5) and introducing the powers of the three systems L_1, L_2, L_3, namely

$$D_1 = \frac{1}{f_1} \qquad D_2 = \frac{1}{f_2} \qquad D_3 = \frac{1}{f_3}$$

we obtain

$$x = \frac{D_1 + D_2 + D_3}{D_3^2} \quad \dots \quad \dots \quad (7)$$

If for example $D_1 = 10$, $D_2 = 20$, $D_3 = 25$ Dp, corresponding to focal lengths of 10, 5 and 4 cm. then

$$x = \tfrac{55}{625} \text{ m} = 8{\cdot}8 \text{ cm.}$$

whilst the magnification is

$$v = \frac{D_3}{D_1} = 2{\cdot}5.$$

Assuming L_2 to be a thin lens, the length of the instrument will be about 34 cm. If the aperture of the eye-piece

be made 4 cm., the tangent of half the subjective angle of vision is $\frac{2}{8\cdot 8}$, and thus the whole angle of vision is about 26°. Hence the true angle of vision is $\frac{26}{2\cdot 5} = 10\frac{1}{2}°$ approx., or the field of view embraces 18 m. at a distance of 100 m. This type of sighting telescope is often employed as a sighting telescope for rifles; for special purposes, as for torpedo-boat service, they are made several metres long. The cross-wire (graticule) may be put either at P' or at P'' (Fig. 104).

Fig. 105 shows an outside view of a sighting telescope mounted on a rifle.

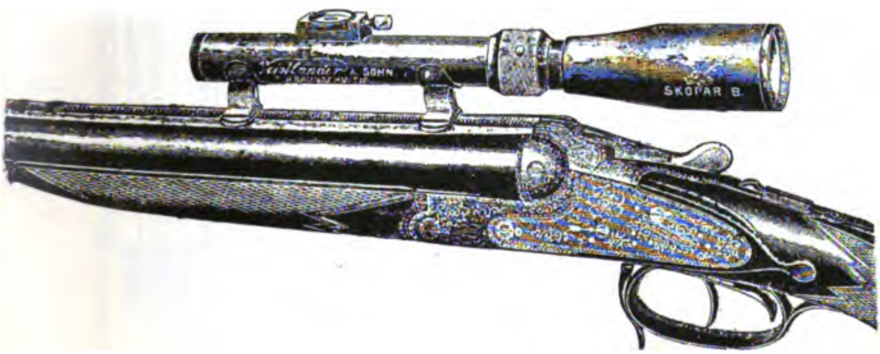

FIG. 105.

The form of the graticule or cross-line diaphragm varies, being often a matter of taste. Figs. 106 to 108 show three forms made by the firm of Busch, of Rathenau.*

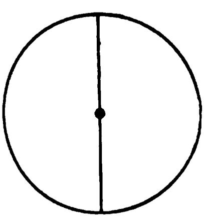
FIG. 106.

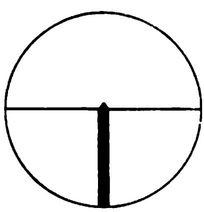
FIG. 107.

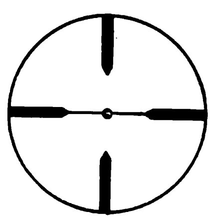
FIG. 108.

These graticules are mounted in a drum and may be brought as required into the focal plane, by a rotation of this drum. In the sighting telescope by Gérard, the objective and the inverting system may be displaced so that the focal plane of the objective may be made to coincide with a second

* Sighting telescopes with interchangeable graticules are described by Weigel in the German Patent No. 170238.

graticule which then comes into view. Since the former graticule now no longer lies in the focal plane, it becomes invisible. In addition there is produced a variation in the magnification corresponding to the above displacement.*

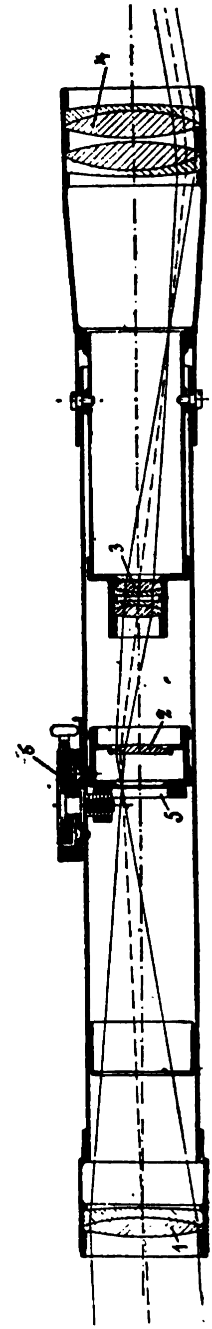

FIG. 109.

The "Certar," made by Goerz, is a modified form of this type of sighting telescope and is shown in Fig. 109. In this figure, 1 represents the objective, 3 the inverting system consisting of two achromatic combinations, and 4 the compound eye-piece. Further, 2 is a field-lens, 5 the position of the graticule, and 6 a screw by means of which the graticule may be raised or lowered for sighting on objects at different distances. It will be seen that on their passage through the inverting system the rays are parallel (*See* § 98).

98. Inverting System having Parallel Rays between its Component Parts.

In the construction of sighting telescopes and similar instruments, an advantage is gained by using an inverting system consisting of two separated lenses between which the rays are parallel.

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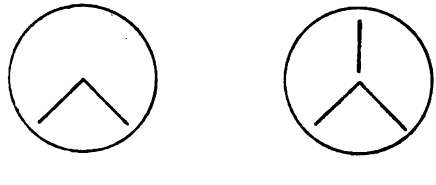
* A form of graticule often used in German instruments, such as panoramic (dial) sights, directors, machine gun sights, &c., is shown in the first figure. The lines are usually etched on a glass disc, but in



some cases (machine gun sights for example) they are made from fine wire bent at right angles and attached to the sliding mount. In some of the German panoramic sights a third line is added as illustrated in the second diagram.—Trans.

In Fig. 110, L_1 and L_2 are the two component lenses (here assumed thin). The objective (which is not shown in the fig.) forms the image $AF=y$, in the focal plane of L_1. From A the principal ray AS passes undeviated through the lens L_1 and strikes the second lens at E, making an angle w with the axis. The extreme rays AB

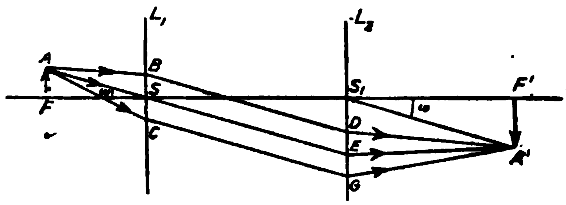


FIG. 110.

and AC from A are refracted by the lens, so that BD and CG are parallel to SE. Consequently they will re-unite after refraction through L_2 in the focal plane of the latter. This focal plane cuts the axis in the second focal point F' of the second lens. In order to find the point A' we draw from the vertex S_1 of the lens L_2 a line parallel to SE and cutting the focal plane in A'. The focal length of L_1 is FS and of L_2 is S_1F'. Further:

$$A\hat{S}F = F'\hat{S}_1A' = w$$

and if $FA=y$, $F'A'=y'$, $FS=f_1$ and $S_1F'=f_2$, we have from Fig. 110

$$w=\frac{y}{f_1} \qquad w=\frac{y'}{f_2} \quad \text{and} \quad \frac{y'}{y}=\frac{f_2}{f_1}$$

The image formed by the objective is consequently magnified by the inverting system by an amount equal to the quotient of the focal lengths of the component lenses. If these two focal lengths are equal, there results no alteration in magnification; only an inversion of the image then occurs.*

99. Best Position for the Line of Sight.

In Fig. 111, A is the direction of the bore of a gun which is assumed to lie to the left of the figure. The curved line represents the trajectory. Suppose now that the gun possesses three different sights which are all sighted,

* Examples showing the application of the above are to be found in the British Patent No. 3744, 1902, describing a submarine telescope by Triulze; in the French Patent No. 370,853 for the observation telescope of Daubresse, and in many modern sighting telescopes, cystoscopes and similar instruments. It will be noticed that the inverting arrangement explained in § 92 could be successfully introduced here.

say, on an object at the distance d_1. The three lines of sight intersect at a point on the trajectory at this distance. The central line of sight is tangential to the trajectory at this point, whilst the other two lines S_1 and S_2 are secants of the same. It will easily be seen that S_1 also cuts the trajectory in front of the striking point and S_2 behind it. According to a publication by Zeiss,

The best line of sight is that which is tangential to the trajectory at the aiming point.

The following considerations will serve to explain this statement. If the aiming point on which the gun-layer has aligned his three sights be exactly at a distance d_1, then all the three lines of sight pass through the striking point, and are consequently equivalent. In general, however, there will be an error in the sighting, and the true distance of the aiming point may be greater (d_2) or less (d_0) than d_1.

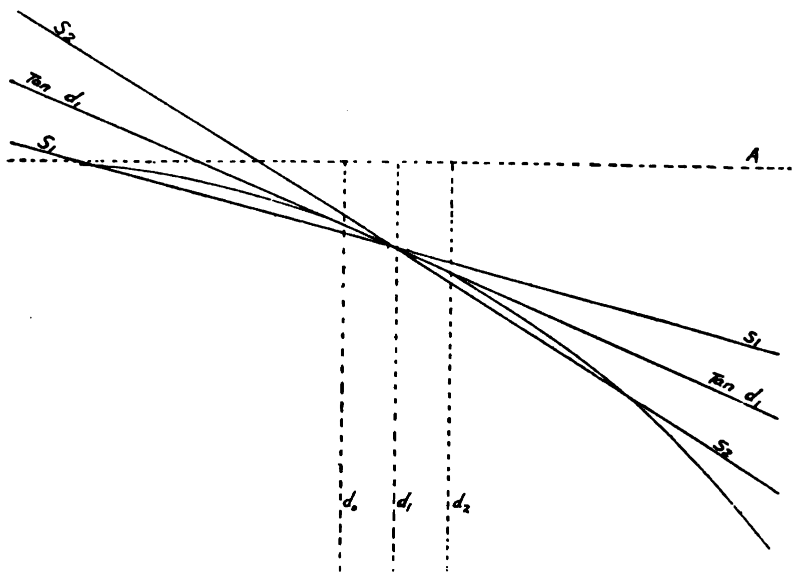


FIG. 111.

If the true distance be d_0 and the line of sight S_1 be used, then the point of intersection of the latter with the dotted line d_0, is the position of the aiming point, and we see from the figure, that in this case the curve passes considerably above the target. If the gun-layer uses S_2, then the aiming point lies at the intersection of the latter line with d_0, and the curve passes considerably under the target. The error for the tangent line of sight, tan d_1 is, however,

comparatively small as a glance at Fig. 111 shows. A similar consideration holds when the error of estimation is in the opposite direction.

In this way we are able to recognize the truth of the above rule.

The firm of Zeiss have introduced a modification of the usual sighting telescope in the form of a prismatic telescope. An important advantage of this construction lies in the elevated position of the objective, as a result of which, in spite of the larger field of view, the barrel of the gun does not obstruct vision. Further, it satisfies to a great extent the above rule as to the position of the line of sight. Figs.

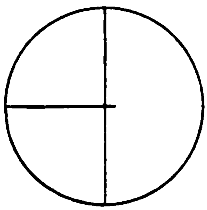


FIG. 112.

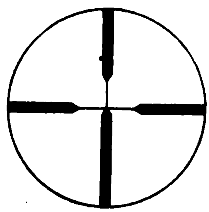


FIG. 113.

112 and 113 show two forms of graticule that are employed. In the particular form shown in Fig. 113, the length of the three thin lines is such that the space covered at the target is equal to $\frac{1}{50}$ of the range. Hence if the size of an object be known, a rough estimation of the range is obtained.*

The sighting telescope for artillery made by Goerz also belongs to this type (Fig. 114). The optical elements

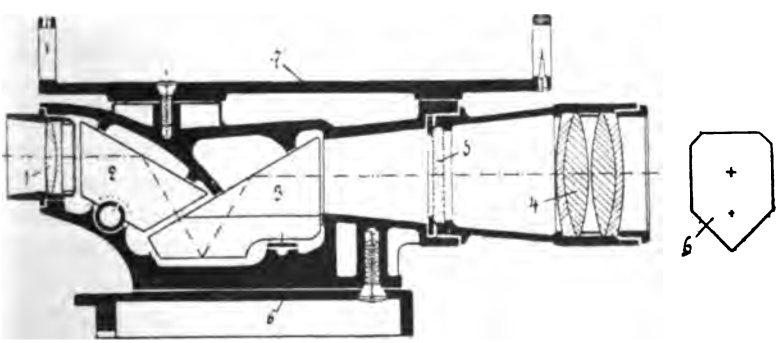


FIG. 114.

of this telescope consist of the objective 1 ; the two prisms 2 and 3 forming the prismatic inverting system ; the

* A further description of the Zeiss sighting telescopes for rifles will be found in the German Patents Nos. 129674 and 133552.

diaphragm plate 5, consisting of a plane parallel sided plate with etched cross-lines, and the eye-piece 4, consisting of two achromatic components. The prismatic inverting system belongs to the group of direct vision Amici-Abbe prisms. The first prism of the system is a 60° prism and the central ray traverses a parallel layer of air between the two prisms, thereby avoiding the necessity of cementing them together. The inverting system is mounted in a rigid aluminium holder provided with projecting ribs to support the prisms. The foot of the instrument is 6, and 7 is a finder with a circular aperture.

100. Panoramic Telescopes in General.* Panoramic Telescope of Goerz.

For purposes of observation as well as for the sighting of targets, it is advantageous if one can survey the whole horizon, or at any rate a greater part of it, without the necessity of rotating the eye-piece to any great extent. This problem arose out of the requirements of the artillery. In gun sighting, it frequently happens that the target is invisible and it is necessary to make use of auxiliary aiming-points which may lie behind the gun. In addition, the necessity arises in the case of submarines, for some optical device for quickly surveying the whole, or the greater part of the horizon.† Around a principal objective are arranged a series of smaller objectives, the images projected by which are observed by means of several eye-pieces or on a suitable screen. Rehm and Windstosser, French Patent No. 360793, in a similar fashion make use of several objectives with different focal lengths, and directed towards different sides, the rays entering the eye-piece after traversing an inverting system, and coming to a focus in one common focal plane. A single eye-piece only is used, and the field of view appears to consist of several images of different sizes corresponding to the various portions of the horizon. The Electric Boat Co., of America‡ have also adopted the method of using a number of objectives directed towards different sides, with eye-pieces arranged in a circular manner.

Another means of viewing the whole horizon, not simultaneously, but one part at a time, is afforded by using an entrance reflector capable of rotation. The entrance

* Generally called "Dial Sights" in the British Services.—Trans.

† One arrangement for this purpose is described in the French Patent No. 368594 (by Simon Lake).

‡ French Patent No. 337549.

reflector consists of a single total reflecting right-angled prism. Now if this reflector be rotated, the reflected image will become inclined, and in order continually to counteract this *leaning* of the image there must be introduced an erecting prism of the form shown in Fig. 77, which rotates at half the angular speed of the entrance reflector. This arrangement is to be found, for instance, in the sighting telescope by Krupp, of Essen*. A second method is also dealt with in this patent, the arrangement consisting of a pentagonal prism in conjunction with an Amici prism. As entrance reflector a three-sided prism placed in front of the objective (or a combination of two such prisms with their bases adjacent) may be used (*see* Fig. 115). The rotating erecting prism, which corrects the *leaning* of the image, may then be dispensed with.†

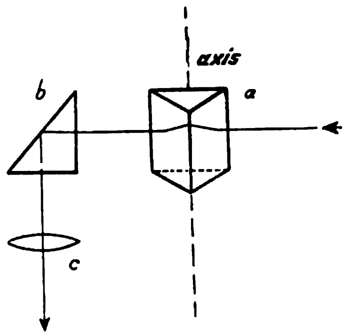


FIG. 115.

In Fig. 115, *a* is the prism (rotating around a vertical axis) which acts as an erecting prism ; *b* is a right-angled totally reflecting prism which reflects the incident horizontal axial ray through 90° into the vertical ; *c* is the objective. If the prism *a* consists of only one prism, then only one half of the horizon may be swept out, say on the left or on the right. In order to survey the whole horizon, two prisms with their bases adjacent must be employed. With the introduction, in the design of modern artillery, of the long barrel recoil carriage in place of the old rigid carriage, the development of optical instruments for sighting purposes has made considerable progress. The instrument is now

* Swiss Patent No. 34870. The use of a rectifying system of the types in question was first described by Sir Howard Grubb in his British Patent No. 5806, 1901, where the application to surveying instruments of the rhomboid prism driven at half the speed of the top prism, as in a panoramic sight, is fully dealt with ; but the credit of designing and applying panoramic sights for artillery and similar service, is due to the energy and skill of Messrs. Goerz.—Trans.

† Such an arrangement is described in the French Patent No. 401021 (Société Huet).

fixed to a part of the gun carriage which does not take part in the recoil and hence is not exposed to severe shock; this is important as the apparatus must have the accuracy of an instrument of precision. Below we give a detailed description of one of the most important examples of this class of instrument, viz., the panoramic sighting telescope of Goerz, of Friedenau (designer, Jacob). There are two methods which afford solutions to this problem of viewing, part by part, the whole of the horizon by means of a rotating optical element and a fixed eye-piece.

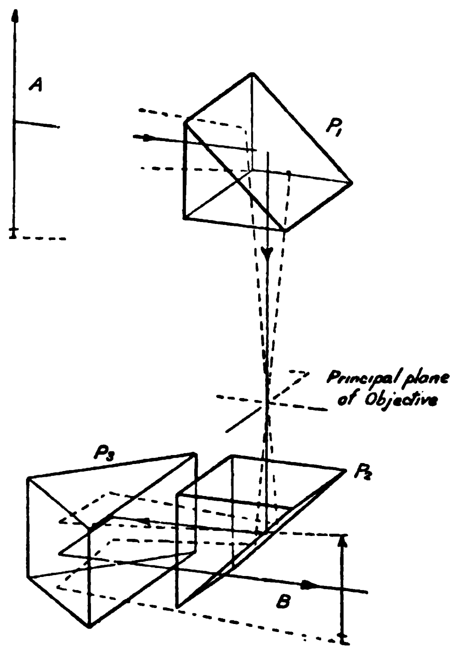


FIG. 116.

1. In the first form (Fig. 116), the three prisms P_1, P_2, and P_3 together form a Porro-system of the first type (if P_1 be conceived to be lowered perpendicularly on to P_2 and placed so that the adjacent faces of the two prisms are parallel). The incident light from a distant object A is totally reflected at the hypotenuse face of the prism P_1, and enters the objective which is situated between P_1 and P_2, after which it emerges, and is totally reflected once in P_2 and twice in P_3. In this way an erect image B is produced in the focal plane of the objective. If now the prism P_1 be rotated around a vertical axis, images are formed of different parts of the horizon in succession. The image, however, does not remain erect, but inclines by an amount depending

on the amount of rotation of P_1. Thus after a rotation through 90°, the image is changed from the vertical to the

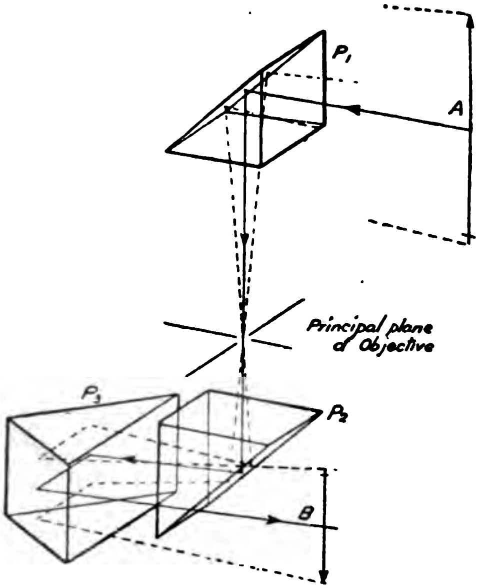


FIG. 117.

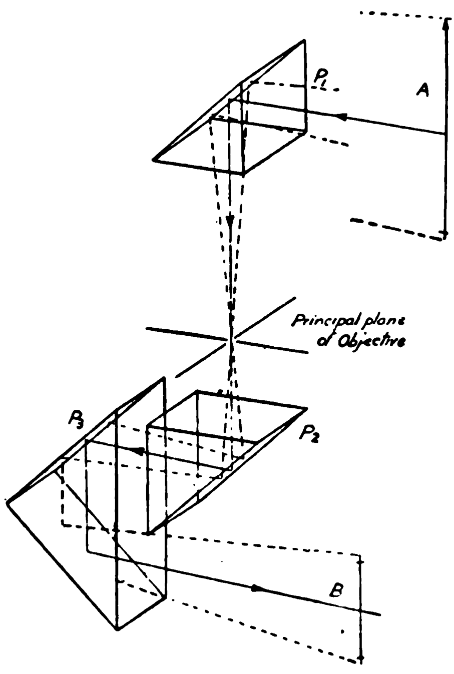


FIG. 118.

horizontal, whilst after a rotation of 180° it is turned upside

down This is technically described as **leaning** of the image. The upside down condition is shown in Fig. 117 where the descriptive lettering has the same significance as in Fig. 116. Now in order to prevent the above leaning of the image, the prism P_3 must be rotated about a *horizontal* axis simultaneously with P_1, and at half the angular speed.

Fig. 118 shows the prisms P_1 and P_2 and the object A in the same position as in Fig. 117, *i.e.*, with the prism P_1 turned through 180° from the position shown in Fig. 116. The image B, however, is no longer inverted as in Fig. 117, but erect as in Fig. 116. This is due to the fact that P_3 has been turned through 90° round a horizontal axis, *i.e.*, one half the rotation of P_1. The actual construction of such an instrument is shown in Fig. 119. The positions of the prisms are as shown in Figs. 116–118. The prism P_1 is

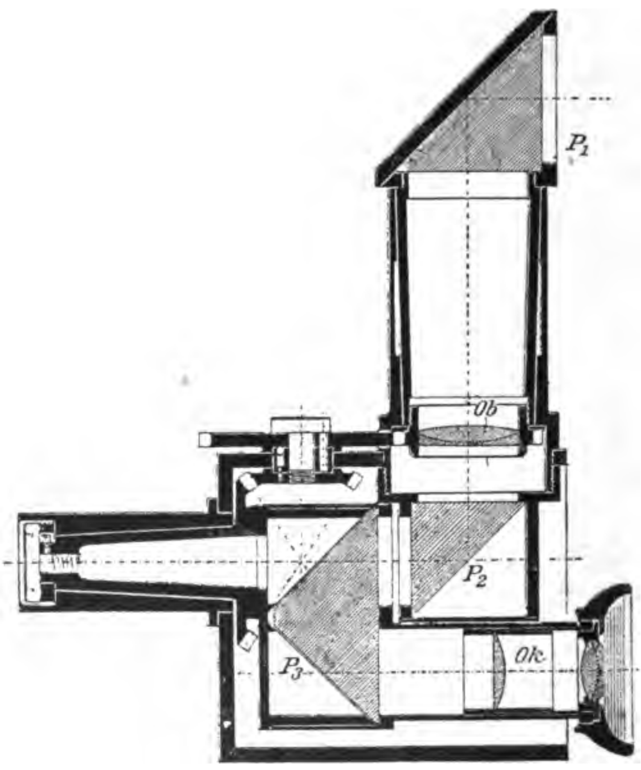


FIG. 119.

mounted in a holder which is capable of rotation, whilst P_2 remains fixed in the body. *Ob* is the objective. The prism P_3 and the eye-piece *Ok* are fixed in a case which can swing round a horizontal axis. By means of bevel wheels

the casting carrying P_1 and that containing the prism P_3 with the eye-piece, are given such a relative motion, that the prism P_3 rotates at half the angular speed of the prism P_1.

2. In the second form of panoramic sighting telescope of Goerz, the leaning of the image due to the rotation of P_1 is prevented by interposing in the path of the rays an erecting rhomboid prism (Fig. 77), which is rotated also at half the angular speed of the prism P_1. This form is shown in Fig. 120.

With regard to the optical arrangement, 79 is the receiving reflector which rotates at twice the speed of the erecting prism 80. The resultant effect of this combination is the same as that of the pentagonal prism described in §87, *i.e.*, the direction of observation may be turned through 90° and still the object appears erect. The parts 79 and 80 must be combined with a terrestrial telescope, or its equivalent, in order to obtain erect images. Thus, for this purpose we may employ an astronomical telescope provided with a Porro-prism system. If it be required to turn the path of the rays through 90° we must use, instead of the Porro-system, a roof prism (Amici prism §88). In Fig. 120, 81 is the objective; 82 the roof prism; and 84 the astronomical eye-piece.

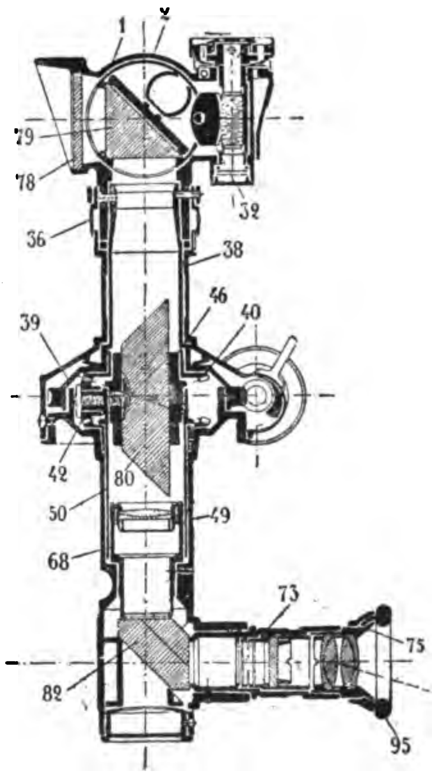


FIG. 120.

The mechanical arrangement is as follows :—

The horizontal limb of the bottom knee piece 49 supports the eyepiece and its holder 75 with shade 95 and cross-wire diaphragm 83, whilst the vertical limb contains the objective tube 68. The cup-shaped extension at the upper end of the bottom knee piece contains the training mechanism. Above this is the spindle case 38, which guides the rotating tube,

to which is fixed a piece 39 carrying a toothed ring 40; on the upper end of 39 rests the reflector holder. The parts 38 and 49 are fixed together with screws. The holder for the erecting prism 80 is fixed in a rotating tube 50 inside the vertical limb of the knee piece. The tube 50 carries the small bevel wheel 42 on a spindle at the side. This small bevel wheel gears with the two large bevel wheels, of which the lower is fixed to the knee piece, and thus incapable of rotation, whilst the upper is fixed to the piece 39. Then if 79 be rotated with the bevel wheel 40 about a vertical axis, 42 rotates with half the angular speed. Hence, since 42 is fixed to the tube carrying 80, the latter will likewise rotate with half the speed, so that the condition that the image remains erect is fulfilled.

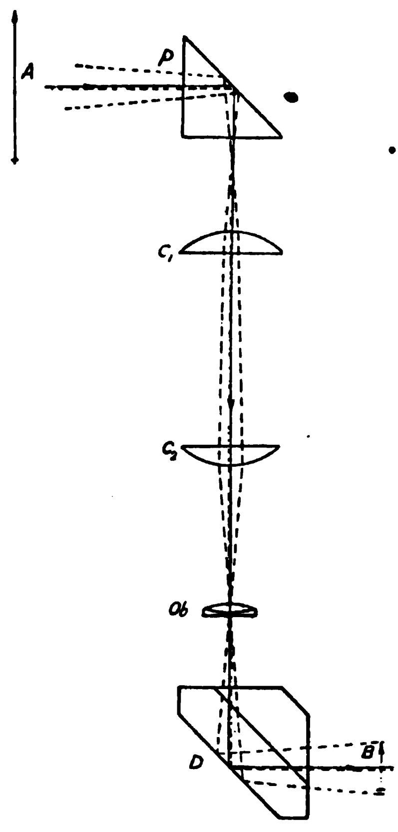


FIG. 121.

Jacob, the designer of the panoramic sight, has found that the erecting prisms of Fig. 120 may be replaced by a combination of two cylindrical lenses. As shown in Fig. 121, two identical cylindrical lenses C_1 and C_2 are introduced between the entrance reflector P and the objective Ob. The axes of these cylinders are parallel to each other and horizontal; *i.e.* perpendicular to the plane of the paper. The common focal plane of these cylinders C_1 and C_2 lies midway between them, and thus the combination acts as an astronomical telescope in which the objective and eye-piece have equal focal lengths; that is with a magnification of unity. Unlike the astronomical telescope, however, there is complete inversion in one plane, and none in a plane at right angles to it, since the convergence of the rays which lie in planes

parallel to the axes of the cylinders is not affected, and consequently no inversion of the image is produced. The system C_1 C_2 thus acts in exactly the same way as the erecting prism in producing a reversal of the image from side to side. If now the cylinder system rotate with half the angular speed of the entrance reflector the erect "attitude"* of the image is maintained.

The distance between the entrance reflector and eye-piece is so fixed that the gunlayer can sight directly behind him over his head. At the lower end of the reflector cowl is a circular scale graduated into 64 parts.†

101. Optical Instruments for Artillery by Zeiss.

The instruments for artillery made by Zeiss are simpler in construction than the panoramic sight of Goerz, at the expense, however, of the property of being able to bring into the field of view all parts of the horizon with a fixed eye-piece.‡

The main purpose of the first of these instruments is to supply the gunlayer with a telescope system that will enable him to sight on two opposite points of the horizon without any change taking place in the direction of the emergent rays; thus making it possible for him to utilize auxiliary aiming-points lying to the rear, without changing his direction of vision. Moreover, the whole horizon may be viewed without changing the direction of observation by more than 90°. Fig. 122 represents one arrangement.

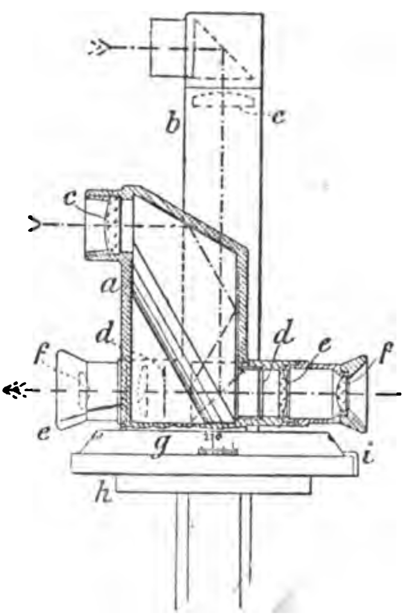


FIG. 122.

The instrument consists of two telescopes *a* and *b* fixed firmly together. In the telescope *a* the axes of the incident

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* Attitude is another term employed to describe the "leaning" of the image.—Trans.

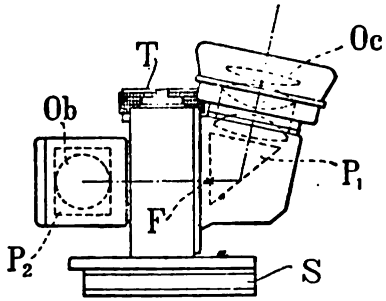
† *See* Arch. f. Optik., Vol. I., p. 241, et seq.

‡ *See* German Patents Nos. 165641, 197105, 221234, 202486.

and emerging rays are in the same direction, whereas in *b* they are in opposite directions. In *b* the axis of the incident beam lies over the head of the observer. With regard to the objective *c*, graticule *d*, field-lens *e*, and eye-lens *f*, both telescopes are equivalent. The graduated circle *g*, which serves as the support of the telescope, is capable of rotating on the head *h* of the instrument base around the vertical axis of the circle. If the instrument be rotated through 180°, the observer, who was previously looking at the landscape straight ahead of him, now sees the landscape lying behind him. In other forms the two telescopes have several optical parts common, and the change over from one position to the other takes place only by a rotation of reflecting systems.

There is another form of instrument in which three telescope systems with a common emergent beam are employed.* Using the telescopes one after the other, it is possible, by means of rotations of less than 90°, to sweep the whole horizon.† In another type of instrument for artillery purposes, and which contains a prism system, the telescope directed towards the front can be turned to the right or left or to the rear by rotating a component system.

The double-direction sighting telescope of Zeiss (Figs. 123–126) is designed for mountain artillery. The direction of the eye-piece axis may vary. In one form, in which the observer looks into the eye-piece almost in the same direction as the line of sight, an astronomical telescope is split into two parts. Between the objective and eye-piece is a simple prism; a rotatable pentagonal roof prism is arranged in front of the objective, so that it can rotate though 180°



FIGS. 123.

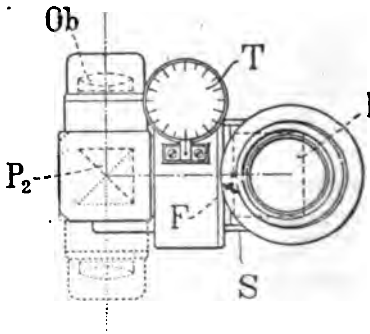


FIG. 124.

about the axis of the objective. In one position the telescope serves for direct sighting; in the other position, and

* German Patent No. 197105. † German Patent No. 202486.

with the help of the graduated circle, for indirect sighting on auxiliary aiming points. In the other type, in which one looks into the eye-piece from above, only two simple prisms are necessary. The letters on the figures have the following meanings.

Figs. 123 and 124 :—*Ob* objective ; P_2 simple rotatable prism ; *F* diaphragm ; P_1 simple fixed prism ; *Oc* eye-piece ; *S* slide for fixing ; *T* driving head for altitude.

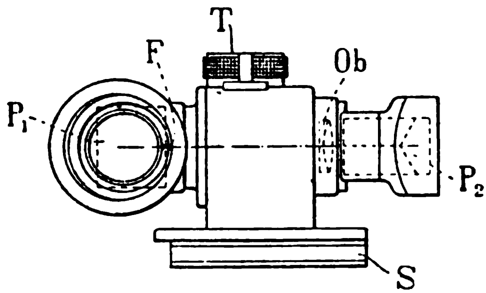


FIG. 125.

Figs. 125 and 126 :— P_2 rotatable pentagonal roof prism ; *Ob* objective ; *F* diaphragm ; P_1 fixed prism ; *Oc* eye-piece ; *S* slide for fixing ; *T* driving head for altitude.

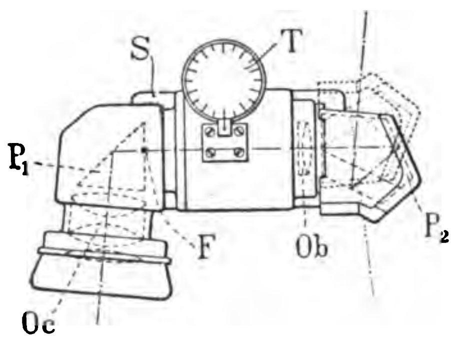


FIG. 126.

The second position of objective and objective prism in Fig. 124 and of pentagonal roof prism in Fig. 126 is shown dotted.

Finally we will mention the arrangement introduced by Zeiss for adjusting the sights of artillery. By means of this arrangement (Fig. 127) the axis of the bore of the gun can be fixed, and the axis of the sighting telescope set parallel to it. An objective whose focal length is equal to the length of the gun barrel, is arranged centrally in the mouth of the gun by means of a suitable plug. At the breech end is a similar plug with cross-wires lying in the focal plane of the objective. The cross-lines are etched on a field lens, so that edge rays, corresponding to the size of the lens, are deviated into the eye, which is placed at the end of the breech ; thus a greater extent of the object is seen. The

whole optical arrangement then represents a telescope whose magnification is equal to the focal length of the objective

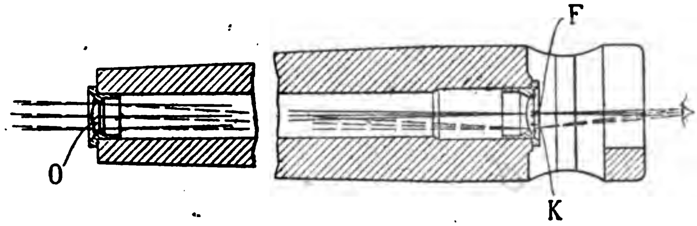


FIG. 127.

divided by the distance of the eye from the cross-lines. The line of sight will coincide very nearly with the axis of the barrel, since small displacements of the objective and cross-lines from their central positions are of little consequence, on account of the large focal length of the objective. To adjust the sighting telescope, it is directed on to the same distant target on which the telescope in the gun is directed. Or we may sight both telescopes on a surface near at hand on which there are two marks whose distance apart and orientation are the same as those between the sighting telescope and the axis of the gun. Or again, we may direct both telescopes on one another by means of a triple mirror (*see* §114). In Fig. 127, O is the objective; F the cross-wires; and K the field-lens.

102. Systems which view the whole Horizon simultaneously.

If in Fig. 128, AB be a portion of a circle, and if this circular arc be rotated about the axis LD lying in the plane of the paper, then the arc AB describes a curved conical surface which may be described as a toric surface. The traces of this surface in the plane of the paper are the arcs AB and EG. If this surface be reflecting, it will form upon a screen placed below it, real images of distant objects on the horizon. With such an arrangement we are thus able to view the whole of the horizon simultaneously. If HJ be a horizontal incident ray, and JM the normal at J, then the ray is deflected to K, making

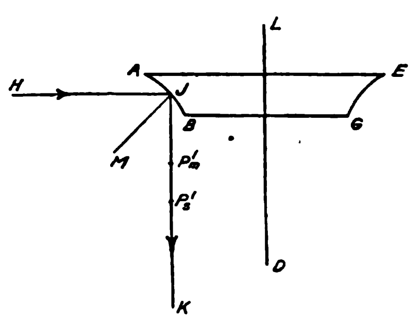


FIG. 128.

the angles $H\hat{J}M$ and $K\hat{J}M$ equal. A very distant object point from which the ray HJ originates will form an image at P'_m by the meridional rays; the sagittal image point is P'_s. From the laws of astigmatic reflection, we have*

$$JP'_m = \frac{r \cos i}{2}$$

where r is the radius of curvature of the arc AB in the meridional plane (plane of paper); whilst

$$JP'_s = \frac{r_1}{2 \cos i}$$

where r_1 is the second radius of the toric surface (produced by rotation) in the sagittal plane. If the astigmatism be eliminated, then $JP'_m = JP'_s$ and therefore

$$\frac{r}{r_1} = \frac{1}{\cos^2 i}$$

This equation represents the condition which a toric surface must fulfil for all points in order that the image is free from astigmatism. If, by the use of this type of toric reflector, astigmatism be eliminated, there still remains in the above arrangement a certain amount of spherical aberration, deviation from the sine condition, and, of course, distortion, which latter cannot be eliminated.

The idea of viewing the whole horizon by means of reflecting or refracting toric elements was first applied by the French engineer Mangin in the construction of a periscope.† In Fig. 129 is shown a meridional section of a toric element used in the Mangin arrangement. The incident light first penetrates normally a spherical surface AB, is then reflected at the paraboloid surface AC, and emerging normally through the spherical surface BC, forms the image point O.‡

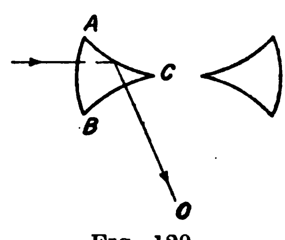


FIG. 129.

* *See* Gleichen, "Lehrbuch der geometrischen Optik," Leipzig, 1902, p. 46, *et seq.*

† *See* Association Française pour l'Avancement des Sciences. 7ème Session, Paris, 1878, p 339.

‡ Recent accounts of this kind of periscope are to be found in

R. d'Equevillez "Untersee-und Tauchboote," Kiel, 1905, p. 57 (Periscope of Mangin).

Maurice Gaget, "La navigation sous-marine," Paris, 1901, pp. 440–444. *See also* French Patent No. 338386.

The periscope of the Improved Periscope Co. differs from the above construction in the use of a hyperboloid in place of a paraboloid; French Patent No. 402467.

CHAPTER X.

Stereoscopy.

Vision through Binocular Telescopes.

103. Fundamental Principles: The Radius of Stereoscopic Vision.

If a body which is not too far distant be observed with both eyes, then in consequence of the distance between the eyes—the **interocular distance**—the observer is able to see to some extent round this body ; in this way a conception of depth of space (or relief) is formed. From the point of view of geometrical optics the explanation for this is, that on different portions of the retina different images of the object are formed.* This phenomenon is called **Stereoscopic Vision.**

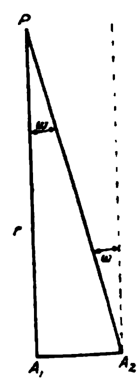


FIG. 130.

In Fig. 130, let A_1 and A_2 be the two eyes of an observer directed on the same point P of an object. We will assume that the distance between the centres of the two eyes is small compared with the distance PA_1 or PA_2. Hence the angle $A_1\hat{P}A_2 = w$ is small, and we may regard the triangle PA_1A_2 as being either right-angled or isosceles.

If we put $PA_1 = PA_2 = r$ and $A_1A_2 = b_0$, we have

$$r = \frac{b_0}{w} \quad \dots \quad \dots \quad \dots \quad (1).$$

The image of the point P formed by the eye A_1 lies on the optical axis, whilst the image formed on the retina of the eye A_2 lies to the right of the axis. As we have seen, the displacement of this image point from the axial point of the retina will be appreciated only when the corresponding angle in the object space is greater than the angle w_0, where w_0 is the limiting angle of resolution of

* See the article on the "Stereoscope" in Ency. Brit. (New Edition) by C. Pulfrich. for a descriptive account of stereoscopic vision.—Trans.

the eye. If the point P be so far distant that $w = w_0$, then stereoscopic vision just ceases. Further, if the corresponding value of r be r_0, then equation (1) gives

$$r_0 = \frac{b_0}{w_0} \quad \ldots \quad \ldots \quad \ldots \quad (2).$$

r_0 is called the **Radius of stereoscopic vision.**

On substituting $b_0 = 65$ mm. and $w_0 = 0{\cdot}000145$ radians (which corresponds to an angle of 30″), then

$$r_0 = \frac{65}{0{\cdot}000145} = 447 \text{ m.} \quad \ldots \quad (3).$$

All points which are further from the eyes than 447 m. cannot be differentiated by them from points on an infinitely distant plane (w_0 has been assumed to be equal to 30″).

104. Radius of Stereoscopic Vision with increased Stereoscopic Base. Relief.

The length b_0, which is called the stereoscopic base and to which the radius of stereoscopic vision is proportional, may be artificially increased. The simplest arrangement is the "Tele-stereoscope" of Helmholtz.

As shown in Fig. 131, this consists essentially of four mirrors S_1, S_2, S_3 and S_4 inclined to the line $S_1 S_4$ at an angle of 45°. S_1 is parallel to S_2, and S_3 to S_4. It will be seen that incident light from a distant object enters the observer's eyes A_1 and A_2 after two reflections. The base S_1 S_4 is considerably greater than the interocular distance A_1 A_2 and the radius of stereoscopic vision is increased in the same proportion. Prismatic telescopes with increased inter-objective distance, as made by Zeiss for example, may be conceived as Tele-stereoscopes in each side of which is inserted a telescope system of magnification ν. All angles in the object space, therefore, will be magnified ν-times. Hence the limiting angle of resolution, w_0 in equation (2), is reduced to $\frac{1}{\nu}$th part and the radius of stereoscopic vision is ν-times greater. The stereoscopic radius S of a binocular

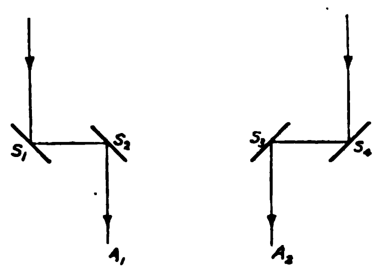


FIG. 131.

telescope with increased objective distance b is therefore proportional to :

1. the quotient of the inter-objective distance and inter-ocular distance, *i.e.*, $\frac{b}{b_0}$;

2. the telescope magnification ν.

Hence

$$S = r_0 \cdot \frac{b}{b_0} \cdot \nu$$

$$= r_0 \, . \, \Pi \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Π is called the **Total Relief.***

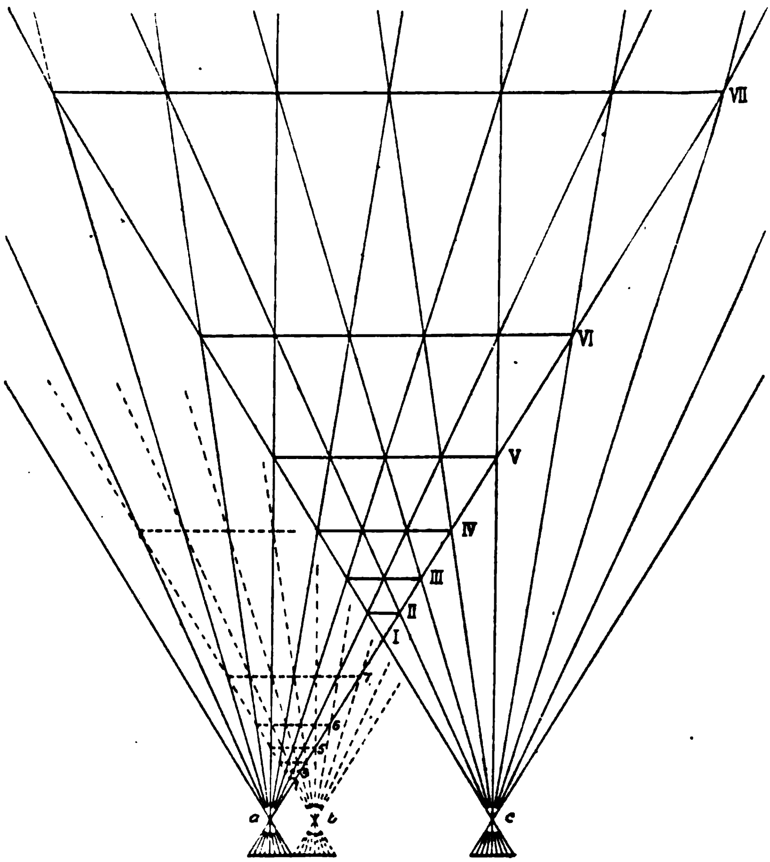


FIG. 132.

* The ratio $\frac{b}{b_0}$ has also been called "base magnification" (by Pulfrich) and "specific plastic." It will be seen in the subsequent paragraph (§ 105) that the stereoscopic effect is determined by the base magnification and telescope magnification. The product of these two, which is here described as "Total Relief," has also been called, Stereo-power."—Trans.

Ex. 101. What is the limiting distance within which stereoscopic vision is possible if a prismatic telescope of 8 magnifications and inter-objective distance of 130 mm. be used?

We have $b = 130$ mm. and $\nu = 8$. Further $b_0 = 65$ mm. and from equation (3), $r_0 = 447$ m.

From equation (4), $S = 447 \times 8 \times \frac{130}{65} = 7152$ m.

105. Vision through Binocular Telescopes.

In vision through a binocular telescope the observer receives the impression of a virtual space image, the dimensions of which are determined essentially by the stereoscopic base (distance between the axes of the objectives) and by the magnification. We will follow here the treatment of Pulfrich.*

In Fig. 132, *ab* represents the base in free vision (*i.e.*, *ab*=65 mm.), whilst *ac* is an increased base, obtained for example, by the use of a Helmholtz tele-stereoscope, in which there is no telescope magnification.

The landscape is visualized as a series of planes; all points on each of these planes being apparently at equal distances from the eyes; these planes, in relation to the base *ac*, are represented by the lines I—VII. The three-dimensional image visualized with reference to the base *ab*, and represented by the planes 1–7, is to be considered as derived from the space image I—VII by a proportional diminution of all three dimensions. The proportional factor is given by the quotient $\frac{ac}{ab}$.

By altering the base alone we merely obtain virtual space images which are similar to the original.

In Fig. 133, we have the case of a binocular telescope of two magnifications retaining the same increased base *ac*. The effect of the magnification is to double all the angles of the central projection; thus in place of the lines I—VII in Fig. 132, appear the lines (1)–(7). These lines, as will be seen from the figure, are moved nearer to the eyes of the observer than the lines I—VII, being only one half as far away, but their frontal dimensions are unchanged.

Hence the increase of telescope magnification alone, (without alteration of base) produces a diminution in the depth dimension proportional to this magnification.

* Pulfrich—"Neue stereoskopische Methoden und Apparate," Berlin. 1903–1909, p. 82 *et seq.*

The opposite holds if the base be decreased and the magnification be less than 1.

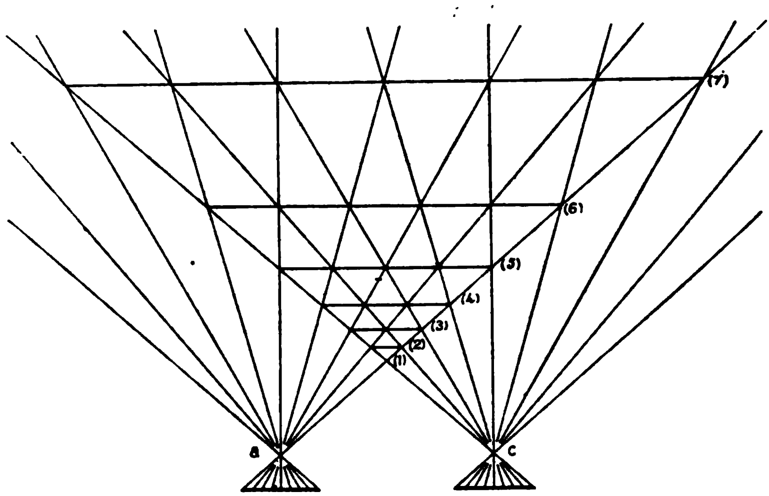


FIG. 133.

The accompanying table by Pulfrich gives a very clear summary of the simultaneous effects of base increase or decrease with telescope magnification or minification.

TABLE (PULFRICH).

Visual Angle.	Normal Inter-ocular Distance.	Instruments which satisfy this condition.	Seen through these instruments, objects of the landscape appear as they would if observed in free vision, their true distance R being replaced by :—	their true dimensions depth × breadth × height being altered from $1 \times 1 \times 1$ to :—
m and n assumed > 1.				
m-magnification.	Unaltered.	Ordinary field-glasses whose inter-objective and inter-ocular distances are equal; looking through eye-piece end.	$\frac{1}{m} \cdot R$	$\frac{1}{m} \times 1 \times 1$
			Frontal dimensions of objects remain unchanged. They appear to be m-times nearer the observer, and depth dimension appears diminished m-times.	
m-minification.	Unaltered.	As above; looking through objective end (*i.e.*, with instrument reversed).	$m \cdot R$	$m \times 1 \times 1$
			Frontal dimensions of objects remain unchanged. They appear to be m-times farther away from the observer and depth dimension appears extended m-times.	
Unaltered.	n-magnification.	Helmholtz tele-stereoscope (without telescope). Stereo-planigraph. Eyes of giant.	$\frac{1}{n} \cdot R$	$\frac{1}{n} \times \frac{1}{n} \times \frac{1}{n}$
			Objects are n-times diminished in all dimensions and appear to be n-times nearer the observer.	

Unaltered.	n-minification.	Eyes of child. Binocular telescope (× 1) with diminished inter-objective distance. (Brachy-stereoscope).	$n \, . \, R$ Objects become n-times magnified in all dimensions and appear n-times farther away from the observer.	$n \times n \times n$
m-magnification.	n-magnification.	$n > m$. Zeiss observation telescope (2m. long; × 10). $n = m$. Helmholtz telestereoscope. Stereoscopic rangefinder. Zeiss relief-telescope (approx.). $n < m$. Zeiss field-glass. $n = 1$. See above.	$\frac{1}{m} \cdot \frac{1}{n} \cdot R$ Objects are diminished n-times in frontal dimensions and appear nearer mn-times; the depth dimension also is diminished mn-times. mn = Total relief.	$\frac{1}{m} \cdot \frac{1}{n} \times \frac{1}{n} \times \frac{1}{n}$
m-magnification.	n-minification.	Binocular telescope with diminished inter-objective distance. (Earlier type of Zeiss opera-glass.)	$\frac{1}{m} \, . \, n \, . \, R$ For $m = n$: Frontal dimensions are magnified n-times; otherwise landscape appears unaltered.	$\frac{1}{m} \, . \, n \times n \times n$
m-minification.	n-magnification.	Child looking through reversed opera-glass.	$m \, . \, \frac{1}{n} \, . \, R$ For $m = n$: Frontal dimensions are diminished n-times; otherwise landscape appears unaltered.	$m \, . \, \frac{1}{n} \times \frac{1}{n} \times \frac{1}{n}$
m-minification.	n-minification.	Giant looking through reversed prismatic field-glass with increased inter-ocular distance.	$m \, . \, n \, . \, R$ Objects magnified n-times in front and appear mn-times farther away. Correspondingly the depth dimension is increased mn-times.	$m \, . \, n \times n \times n$

106. Recent Stereoscopes and Stereoscopic Cameras.

If an object be drawn or photographed twice, once as seen by the right eye of an observer, and once as seen by the left eye, we have what is called a stereoscopic picture. If this latter be observed by a pair of eyes, each eye seeing its corresponding picture, the observer receives the impression of the solidity of the object. Apparatus designed for the observation of stereoscopic pictures are called stereoscopes of which many forms have been developed. The latest design by Zeiss is equipped with magnifying glasses of focal length 15 or 10 cm.; there is provided an adjustment for inter-ocular distance and also for focus, rendering stereoscopic vision comfortable. In order that the eyes may receive the impression of objects in their natural sizes, the glasses must have focal lengths equal to those of the objectives of the camera with which the stereoscopic pictures were taken. Most stereoscopic cameras are provided with objectives of focal length 10 to 15 cm., so that the Zeiss apparatus is suitable.

It is a well-known difficulty in stereoscopic photography, that the two prints obtained from the negative must be separated and interchanged in order to obtain a positive which gives the desired stereoscopic effect. Many attempts have been made to overcome this difficulty;* a new method has been proposed by Fricke.† He suggests that a roof prism of the Amici type be inserted immediately behind the

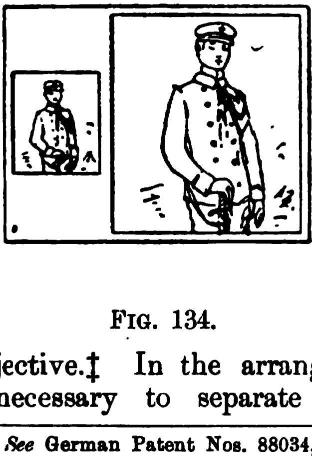


FIG. 134.

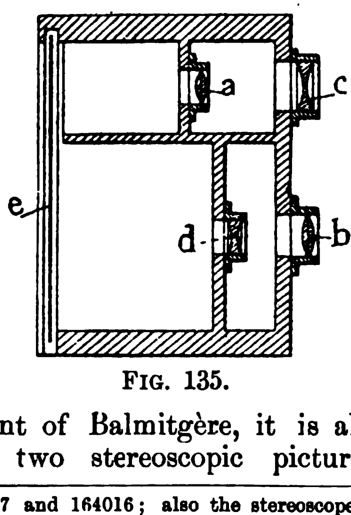


FIG. 135.

objective.‡ In the arrangement of Balmitgère, it is also unnecessary to separate the two stereoscopic pictures.

* *See* German Patent Nos. 88034, 156427 and 164016; also the stereoscope of Steinhauser described by von Stolze in "Die Stereoskopie und das Stereoskop," 1894, p. 119.

† *See* "Zeitschrift für wissenschaftliche Photographie," Vol. 5, 1907, p. 205.

‡ Essentially the same idea is given by Richard in the French Patent No. 396364.

Figs. 134 and 135 show his camera, and Figs. 136 and 137 the corresponding stereoscope. Pictures of different sizes are

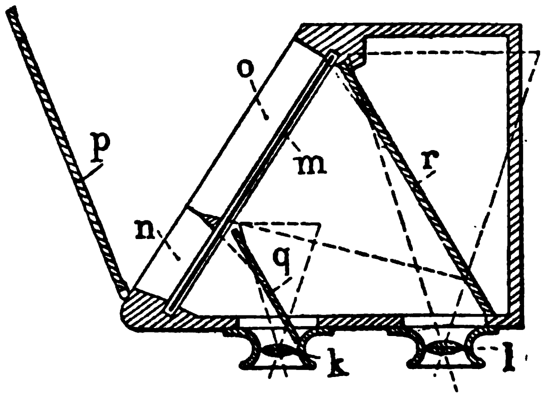


FIG. 136.

formed on one and the same plate by the camera; these are brought together and made to appear the same size by the stereoscope. In order to obtain the two photographs in different sizes on the same plate *e*, Balmitgère uses two combinations, each consisting of a positive and a negative lens; one has the effect of a tele-objective and the other the reverse effect. In the stereoscope, the inversion of the pictures is effected by the inclined mirrors *r* and *q*, so that the pictures at *n* and *o* are respectively observed by their corresponding eye-pieces, *k* and *l*. The pictures are held by the plate *m*, and are illuminated by means of the reflector *p*.

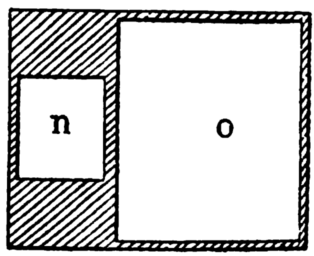


FIG. 137.

CHAPTER XI.

Rangefinders.

107. Rangefinders. General Remarks.

For a long time astronomy and geodesy provided trustworthy methods of measuring distances, all these methods depending on the theorem in geometry that a triangle is determined by two angles and one side.

In Fig. 138 let AC represent a distance, whose magnitude is determined by direct measurement, and which will be called the **base**. B is a point whose distance it is required to find. If B is sighted from the two positions A and C, then the angles α and γ will be determined, and since AC is known, the distances AB and CB are easily found by trigonometrical calculation. If the triangle ABC be isosceles or right angled, then it is necessary to determine only one angle. Usually the base is horizontal, but it may be arranged vertically as in the case of rangefinders for certain military purposes, where the instrument is set up on some high position on the sea coast.* The distance from the instrument to the sea-level is then taken as the base.

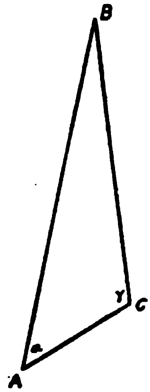


FIG. 138.

For many years manufacturers have concentrated their efforts on the problem of devising for the army and navy a rangefinder which is both comparatively small and as easy as possible to handle.† We will here confine ourselves to a description of the modern types that have been adopted for military and naval purposes.

In Fig. 139 let S_1 and S_2 be two mirrors at a distance b (the base) apart, and inclined to the latter at an angle of 45°. The incident light from a distant object point B is reflected by S_1 and enters the objective L_1; after

* Depression Rangefinder.—Trans.

† Instruments of the earlier type may be found collected and fairly completely described in:

De Marre, "Les instruments pour la mesure des distances." Paris, 1880.

For more recent literature *see* patent specifications.

See also "Ausgewählte Kapitel der Technik," by Niesiolowski-Gamin. Vienna, 1908. The chapter on Rangefinders.

refraction through the latter it is reflected into the eye-piece N by the mirror K_1 which is parallel to S_1. Similarly, light from B enters the eye-piece N after reflection at the parallel mirrors S_2 and K_2 and refraction through the objective L_2. If the mirrors K_1 and K_2 lie one above the other, the light from S_1 illuminates, say, the lower portion of the field of view, and that from S_2 the upper portion. Suppose now that both objectives have the same focal length ϕ and that the base subtends at B the angle w.* We may consider the very narrow triangle BS_1S_2 as being a right-angled triangle. The beams of light forming the two images are reflected at the reflectors S_1 and S_2. The image of B formed by the left objective L_1 will lie exactly on the optical axis; the image of B formed by the right objective L_2 will lie to one side of the axis, its distance from the axis being given by

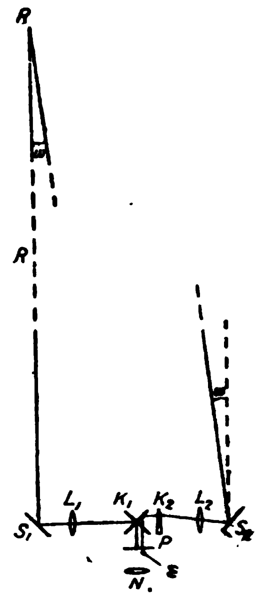


FIG. 139.

$$\varepsilon = \phi\, w$$

Now from the triangle BS_1S_2, if the distance of the point B from the rangefinder be R, we have

$$w = \frac{b}{R}$$

and from these two equations

$$\varepsilon = \phi\, \frac{b}{R}$$

The displacement ε is thus inversely proportional to the range R, and directly proportional to the base b.

Hence in order to determine the range R, we must measure in some way the displacement ε. In practice this measurement is effected by observing the change in position of an optical element which brings the two images of B into superposition or coincidence. This optical element may conveniently consist of a glass prism P of small

* This angle subtended by the base at the object is called the *Angle of Parallax* or *Parallactic Angle*.—Trans.

refracting angle, placed in the path of the rays and travelling parallel to the optical axis. A reading can then be taken on a scale which is fixed to the prism and which moves past a fixed pointer.

The particular division of the scale opposite the pointer then gives the range. The first practicable instruments constructed were those by the English professors, Archibald Barr and William Stroud, followed later by those of Hahn of Cassel, Zeiss of Jena, and Goerz of Friedenau, &c.

The plane mirrors are generally replaced by reflecting prisms. A great number of improvements have been introduced in order to perfect the optical performance, to make the measurement as far as is possible independent of external influences (such as shocks, temperature variations), and to be able to re-adjust the instrument quickly if for some reason it becomes deranged.

108. Coincidence and Invert Systems.

Plate I. Fig. 1 and Fig. 2, show the view as seen through the eye-piece of a rangefinder of the kind described above. The field of view is divided into two parts by a horizontal straight line. Fig. 1 shows the field of view of a **Coincidence** rangefinder. It will be seen that the upper part of the field of view appears displaced a certain amount, as indicated by the top of the mast. In order to obtain a measurement of the range, the two parts of the field of view are brought into coincidence, so that they appear in correct alignment, forming a complete picture.

Fig. 2 represents the **Invert** system. The upper part of the field of view is exactly the same as the lower except that it is inverted, the appearance being as if reflected by a plane mirror. In the case of portions of the landscape with projections clearly defined against the sky, *e.g.*, a tree as in Fig. 2, it is possible to attain great accuracy by this method. The measurement is made by accurately placing the corresponding portions of the landscape (clearly defined projections if possible), into correct alignment or coincidence.

109. Limiting Angle of Resolution.

The smallest object that can be distinguished by the eye subtends at the eye an angle which is called the **limiting angle of resolution of the eye.** Two stars which subtend this angle

would be just distinguished as separate stars by the eye. This angle depends on the keenness of vision S; it decreases as S increases and *vice versâ*. Usually it is taken to be about one minute of arc; that is, calling the limiting angle of resolution of the eye w_0,

$$w_0 = 0{\cdot}000291 \text{ radian.}$$

More recent investigations into the limits of error of measurement, especially those on the stereoscopic principle, have, however, established a considerably smaller value for w_0; and perhaps the most favourably accepted value is 10 seconds of arc. The size of the image on the retina which corresponds to a distant object under the visual angle w_0 is given by

$$y' = f.\, w_0$$

where f is the first focal distance of the eye. For $f = 15$ mm. and $w_0 = 0{\cdot}00029$ radian, we obtain

$$y' = 0{\cdot}0045 \text{ mm.}$$

This gives the extent of the retina which cannot quite distinguish two separate light disturbances (*see* Ex. 56 and § 66).

110. Errors of Observation of Base Rangefinders.*

An elementary description is given below of the method of reckoning errors in the case of base rangefinders, following the summary of formulae and "errors of observation" published by Zeiss for military use.

In Fig. 140 let b be the base which subtends the angle w at the target. Since the base b is very small compared with the range R of the object, the triangle formed by the base and the lines drawn from the extremities of the base to the object may be regarded as isosceles or right-angled; for simplicity we will assume the latter. Hence

$$\tan w = \frac{b}{R}$$

or since w is very small,

$$w = \frac{b}{R} \cdot 206{,}000'' \qquad \ldots \qquad \ldots \qquad (1)$$

---

* By the term *Base Rangefinder* (German—Basisentfernungsmesser) Dr. Gleichen probably refers to all types of rangefinders depending on the measurement of the parallactic angle subtended by the base. The error referred to in this paragraph, depending on the aligning power of the eye in combination with the instrument, applies to mekometers and those instruments in which the height or length of the target itself forms the base.—Trans.

For example, if $b = 1$ m. and $R = 1$ Km. $= 1000$ m., then

$$w = 206'' = 3'\ 26''$$

Neglecting for the moment the telescope magnification of the instrument, there may be an error in the measurement corresponding to some angle f. This error f represents the smallest angle that can be appreciated when using such an instrument. It depends not only on the limiting angle of resolution of the eye (*see* § 109) but is also affected by other influences such as the nature of the object, illumination, optical adjustment of the rangefinder, etc. Since this error may either increase or diminish the correct angle w, the angle actually measured is

$$w' = w \pm f.$$

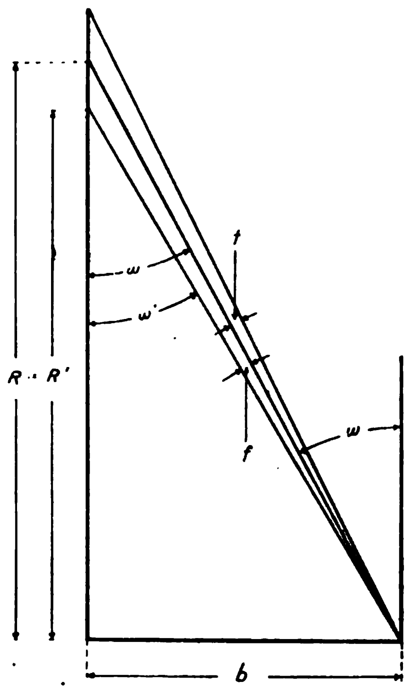


FIG. 140.

On account of this angular error, there will be a corresponding error in the range R' obtained, which is given by

$$w' = \frac{b}{R'} \cdot 206000 \qquad \ldots \qquad \ldots \qquad (2)$$

From equations (1) and (2) we obtain

$$w' - w = \pm f = b\left(\frac{1}{R'} - \frac{1}{R}\right) 206000$$

$$= 206000\ b \cdot \frac{R - R'}{RR'}$$

But since R and R' will differ from each other only by a very small amount, we may write R^2 for the product RR' and thus obtain from the last equation

$$R - R' = \frac{\pm f R^2}{206099\ b} \qquad \ldots \qquad \ldots \qquad (3)$$

Hence the error $R - R'$ in the determination of the range is directly proportional to f and the square of the range, and inversely proportional to the base.

Taking into consideration now the magnification v of the rangefinder, the retinal image of an object subtending the visual angle f is magnified v times, so that the resolving power is increased v-fold. Hence instead of the angle f, we substitute $\frac{f}{v}$. Equation (3) then becomes

$$R - R' = \frac{\pm f R^2}{v\, b} \cdot \frac{1}{206000} \quad \ldots \quad (4)$$

Ex. 102.

Suppose the error f to be 20 seconds. What is the error of observation if $b = 0 \cdot 5$ m., $v = 8$, $R = 1000$ m.?

From equation (4)

$$R - R' = \frac{20 \times 1000^2 \times 2}{8 \times 206000} = 24 \cdot 2 \text{ m}$$

Uncertainty of Observation of Base Rangefinders.

Assuming the resolving power in free vision to be 10 seconds; *i.e.*, under particularly favourable circumstances.

Base in metres	0·50	1·0	3·0
Magnification	8	11	28
Range in metres.	Error in metres.		
200	0·5	—	—
300	1·1	—	—
500	3·0	—	—
1000	12·0	4·4	0·6
2000	49	17	2·3
3000	109	39	5·5
5000	—	110	15
7000	—	218	29
9000	—	358	47
15000	—	—	129

111. Rangefinders of Hahn & Co.

Messrs. A. and R. Hahn, the predecessors of Messrs. Hahn, Ltd., were the first in Germany to apply themselves to the manufacture of monocular rangefinders for military and naval purposes. Their efforts date back to the seventies of the last century, and their instruments were the first to

be supplied to the German infantry and machine-gun corps. Their later rangefinders are made on the Coincidence as well as the Invert principle (*see* § 108). The Hahn infantry rangefinder is shown diagrammatically in Fig. 141.

The instrument consists of two prismatic telescopes the entrance windows of which are at the ends of the base (*i.e.*, 80 cm. apart); the telescopes have an eye-piece common to both. The right-hand prism L_1 yields an erect image, whilst the left gives an inverted image (invert system). Neither image is reverted, that is reversed sideways.

The entering rays are reflected first by the pentagonal prisms P_1 and P_2, and after traversing the objectives O_1 and O_2, are again reflected by the right angled eye-piece prisms L_1 and L_2. The latter are placed one above the other so that the rays of the right hand telescope form the image in the lower half of the field of view, whilst rays of the left hand telescope produce the image in the upper half. In order to erect its image, the lower prism takes the form of a roof prism. The two real images thus produced are magnified by the eye-piece A.

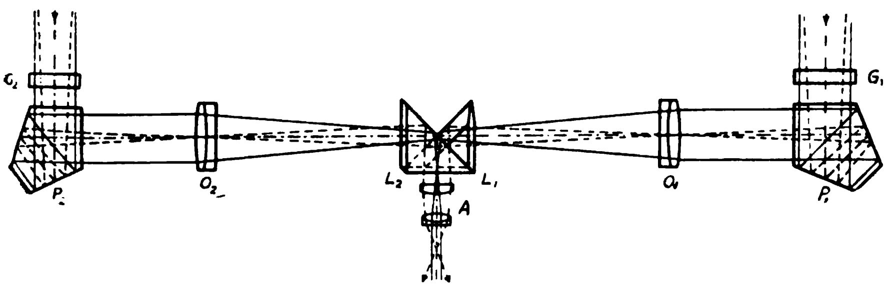


FIG. 141.

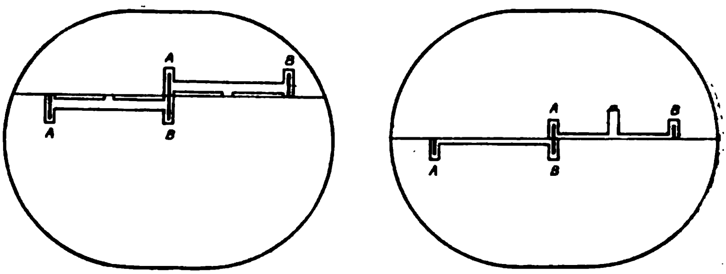
The eye-piece prisms L_1 and L_2 are so constructed that the full aperture of the objective can be used for all points of the field of view. To obtain this result the prisms are provided with sloping entrance and exit surfaces.* The instrument is so adjusted by suitable disposition of the eye-piece prisms and objectives that the exit pupils of both telescopes coincide.

The measurement of the parallactic angle is effected by the movement of an optical element—either one of the objectives or the eye-piece prisms—perpendicular to the

* The reason for the inclined faces does not appear to be clearly described. The faces in question are set normally to the beams of light which are inclined to the longitudinal axis of the instrument owing to the objectives being situated on the axis and the eye-piece prisms above and below the axis respectively.—Trans.

path of the rays. In the arrangement shown in Fig. 141, the measurement is made by moving the objective O_1 along a guide set perpendicularly to the optical axis. Its displacement is brought about by a rotation of the working head, and at the same time the range-recording drum is moved by an amount corresponding to the displacement of the objective. There are provided spiral grooves on the outside of the range drum, and a pointer, moving with a horizontal bar, slides in these grooves, so that the range is read directly on the range drum.

The adjusting or "setting" of the instrument is effected by means of **adjusting laths** (Fig. 142). The lath consists of a tube supported by two legs; the tube carries a small telescope and two vertical marks whose distance apart is equal to the base length of the instrument to be adjusted. It is arranged parallel to the rangefinder at a distance of 80 to 100 m. The legs are adjusted until the cross-lines of the lath telescope coincide with the head of the



(a) UPPER IMAGE INVERTED. (b) BOTH IMAGES ERECT

FIG. 142.

rangefinder stand. The scale pointer is then made to coincide with a certain mark on the recording drum. If the instrument be correctly adjusted, the two images of the lath appear in the field of view of the rangefinder in the relative positions shown in Fig. 142. In this figure (a) represents the field of view of an erect-image rangefinder, and (b) that of a rangefinder in which the upper image is inverted. The introduction of the adjusting lath is due to Messrs. Barr and Stroud.*

112. Rangefinders by Zeiss.

1. The System of Measurement.

The construction of rangefinders was commenced by Zeiss at the instigation of the engineer, Hector de Grousilliers,

* *See* Brit. Patent No. 9520; 1888.

of Charlottenburg, who in 1892 suggested to them the underlying idea of a stereoscopic rangefinder, the practical development of which has been carried out by Pulfrich.

About 1905 this firm turned its attention to the construction of single eye-piece rangefinders ; in the same year they took out a patent* for a particularly simple and stable eye-piece prism system and also a rangefinder on a new arrangement of field—the **Reverted or Side-by-side System.** In this system the field of view is separated into two equal parts by a vertical separating line, and the two images of the object under observation are brought together into symmetrical positions on the right and left of the separating line respectively. The **Invert System** represented in Plate I, Fig. 2, which is a modification of the coincidence system, was first used by Zeiss ; it is particularly suitable for irregular targets such as are found in field work. The practicability of this construction had, however, already been pointed out by the English professors Barr and Stroud,† who have made such great progress in the manufacture of rangefinders. On the discovery of the priority of the English patent, the firm Zeiss withdrew their patent notification.

II. Stereoscopic Rangefinders.

The stereoscopic rangefinder of Zeiss may be considered as a binocular telescope with increased inter-ocular distance ; in each of the two focal planes, one for the left eye

FIG. 143.

* German Patent No. 175900.
† English Patent No. 1462, 1903

and one for the right, is inserted a scale; these scales are so constructed that they unite to form a stereoscopic image in binocular vision. This image then appears to be projected into space and the observer receives the same impression as he would if the pictures shown in Fig. 143 were observed through a stereoscope.*

The scales may be marked on transparent glass plates, with parallel faces. If the scale, as it appears projected into space, be brought up to the object whose range is required, the latter may be read off directly. The scale is made zig-zag for practical reasons.

The superposition of the two scales to form the stereoscopic image will be understood more clearly from Fig. 144.

Let L and R represent the eyes of an observer. The plane containing the scale marks and the image plane, both of which coincide with the focal plane of the binocular telescope, are shown separated in the diagram for the sake of clearness. The marks m'_1, m''_1, etc., in the left hand focal plane correspond to the marks m'_2, m''_2, etc. of the right hand focal plane. These marks unite stereoscopically to form the points m', m'', etc. in space. The intercepts P_1 Q_1 and P_2 Q_2 are the traces in the image plane of the object PQ which is under observation and which appears to lie in the region of the points m', m'', etc.

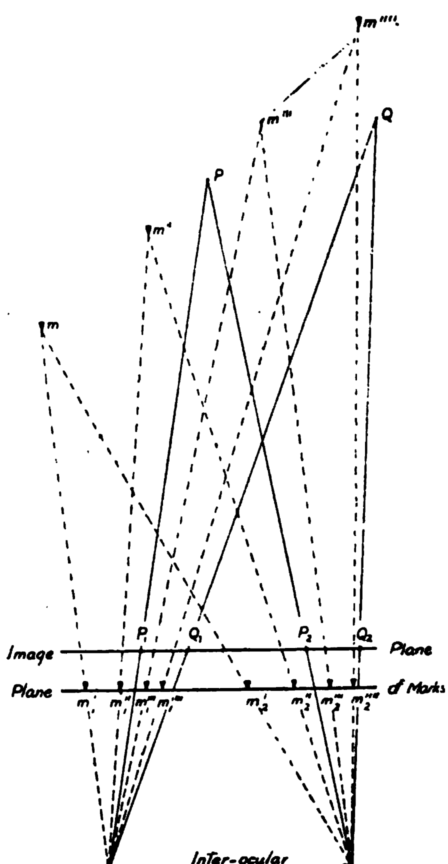


FIG. 144.

Equation (4) may be used to determine the errors of observation; in both cases—that is in the use both of the

* *See* the German Patent No. 82571; 1893.

stereoscopic rangefinder and the monocular instrument—the angular error f depends on the limiting angle of resolution of the eye, so that the table in § 110 is directly applicable to stereoscopic rangefinders.

The expression

$$R - R' = \frac{f R^2}{v\, b} \cdot \frac{1}{206000}$$

may now be written in another form. Remembering that

$$\Pi = \frac{b}{b_0} \cdot v$$

is the **Total Relief** of the rangefinder, where b_0 is the interocular distance, we have

$$R - R' = \frac{f R^2}{b_0\, \Pi} \cdot \frac{1}{206000}.$$

Assuming that the conditions of observation are favourable, then f is equivalent to w_0, the limiting angle that the eye can resolve, and we have

$$\frac{b_0}{f} = r_0$$

where r_0 is the radius of stereoscopic vision with the unaided eye. Hence

$$R - R' = \frac{R^2}{S} \cdot \frac{1}{206000}$$

$S = r_0\, \Pi$ represents the radius of stereoscopic vision obtained with the rangefinder.

Thus the error or uncertainty of observation of a stereoscopic rangefinder is proportional to the square of the desired range and inversely proportional to the radius of stereoscopic vision of the instrument.

III. Second Form of Stereoscopic Rangefinder.

The expression for the stereoscopic radius of a binocular telescope :—

$$S = r_0\, \Pi$$

where

$$\Pi = \frac{b}{b_0} \cdot v$$

is the total relief, has a more general interpretation ; it holds for all ranges R lying within the stereoscopic radius, if by

r_0 is understood not just the maximum range of stereoscopic vision with the unaided eye, but more generally the distance corresponding to any divergence of the beams of rays proceding from the object under observation and entering the two objectives. Thus the apparent range of points in the landscape in vision through a binocular telescope depends on the total relief of the instrument. This is the principle under-

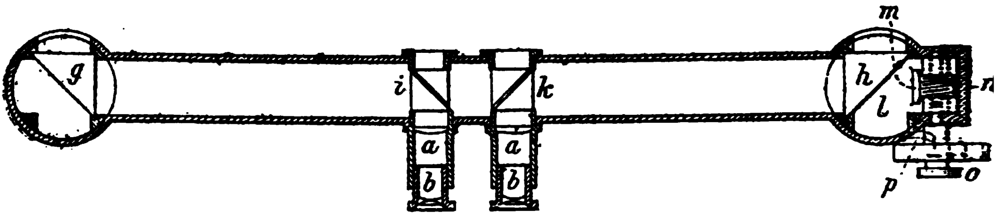


FIG. 145.

lying a recent form of stereoscopic rangefinder by Pulfrich and König.* Two binocular telescopes with different total relief are so combined that the observer can perceive the two space images formed of each object point by the two telescopes, simultaneously, or one after the other in binocular vision. A micrometer arrangement is provided so that one or more of the four systems of beams of rays traversing the two binocular telescopes, can be deviated until the two space images appear to be equi-distant. The range of the object is then read from the micrometer scale.

One form of this rangefinder is shown in Fig. 145. At g and h are two totally reflecting prisms; i and k are two cubic prisms each of which consists of two isosceles right-angled prisms arranged with a space between them; they are half transparent and half reflecting; the objectives are represented by a, and the eye-pieces by b. Thus, if the upper parts of the prisms are transparent, then in the upper half of the field of view the eyes perceive the landscape under a total relief of v, since $b=b_0$. If the lower half be reflecting, the total relief is $\frac{b}{b_0}.v$, where b is the distance apart of the two prisms g and h. In order to make the apparent distances of the portions of the landscape under observation appear equal, the seating l of the prism h is made capable of rotation about a vertical axis; it is provided with a toothed sector m which gears with the measuring screw n. To the latter is fixed the range drum o provided with a scale

* German Patent No. 162471.

which moves past the fixed pointer p. (For further particulars *see* Patent literature. The instrument was not adopted practically.)

IV. Monocular Rangefinder with fixed Scale.

A new design of rangefinder by Eppenstein was brought forward by Zeiss in 1908.* Either the invert or coincidence system may be applied in this instrument. It will now be described with reference to Fig. 146.

A and *B* are two telescopes arranged parallel to one another; the objective in *B* has a shorter focal length than that in *A*. The focal planes of the objectives are represented by $a_1\ a_2\ a_3$ and $b_1\ b_2\ b_3$; the eye-pieces are not shown. The two parallel rays 1 and 1′ form the image points a_1 and b_1. If now the two image points a_2 and b_2 are chosen so that

$$a_1\ a_2 = b_1\ b_2$$

then the rays 2 and 2′ originate from a point at a finite distance. The two rays 3 and 3′ determined by

$$a_2\ a_3 = b_2\ b_3$$

come from a still nearer point.

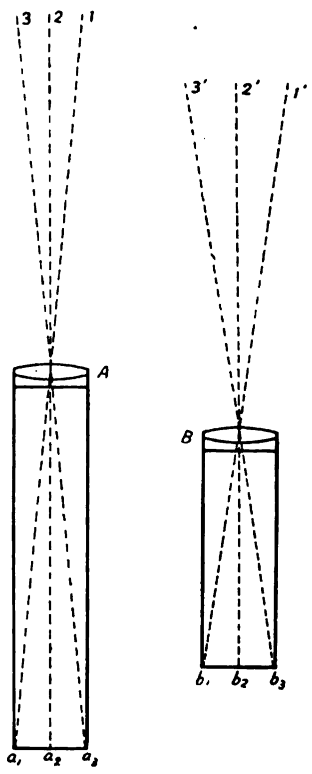


FIG. 146.

Now suppose the two fields of view be separated by a horizontal line and placed contiguous to one another at this separating line as in the coincidence system; the image from *A* lying above and that from *B* below the separating line. If the coincidence rangefinder thus obtained be so adjusted that a_1 is in coincidence with b_1 on the separating line, and consequently a_2 with b_2, and a_3 with b_3, we have an arrangement whereby the images of objects at various ranges are in coincidence. The points on the separating line at which coincidence is obtained, however, vary with the range of the object. Hence if a scale be inserted in the focal plane, the range may be read off directly.

* German Patent No. 205128; 19th Jan., 1908.

V. The Zeiss Rangefinder Adjuster.

An estimate of a range given by a rangefinder is dependent on the correct setting of many parts, and small changes embracing very small displacements of these parts produce serious errors; hence some arrangement was very soon found to be necessary whereby the instrument may from time to time be tested and if necessary adjusted correctly.

The employment of adjusting laths usually takes an appreciable length of time, and is not possible without some preparation; in addition, it involves the carrying of a separate piece of apparatus along with the rangefinder. Hence the firm of Zeiss introduced the "**Internal Adjuster.**"*

These adjusting arrangements, which consist of certain optical parts, are either attached to the rangefinder or are fixed permanently inside it; they assume different forms as follows:

A. Those which depend on the characteristic property of a pentagonal prism, that its angle of deviation is constant.

1. *The Abbe Adjuster*† consists of two pentagonal prisms with their transmission faces turned towards one another as in Fig. 147. We can illustrate its use best by considering it in conjunction with a stereoscopic rangefinder, although its use is not limited to such a rangefinder.

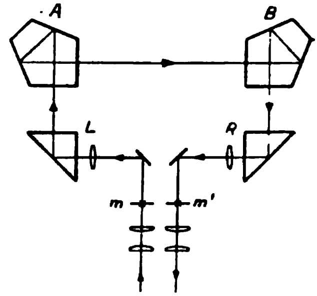


FIG. 147.

Consider a source of light placed behind the left eye-piece; the rays will traverse the left half of the rangefinder. A bundle of rays from one of the stereoscopic marks m will leave the objective at L as a parallel beam; it will then be reflected in the two pentagonal prisms and become incident on the objective at R, producing in the right hand focal plane an image m' of the mark m. If this image m' coincide with a mark n in the right eye-piece, then the marks m and n become stereoscopically united, appearing to be at a distance away equal to that of the point of intersection of the rays L and R. This distance depends only on the sum of the angles of deviation of the pentagonal prisms. If this sum be 180°,

* This should not be taken to mean that Internal Adjusters were first introduced by Messrs. Zeiss. Many of the adjusters described were proposed and used by Professors Barr and Stroud at an early stage in the history of rangefinders.—Trans

† German Patent No. 73568; 1893.

then L and R are parallel, and the marks m and n, which we then have to consider, are those which would appear at infinity when looking through an instrument which is in correct adjustment.

Hence the rangefinder may be tested by looking through the right eye-piece and observing whether the left infinity mark is imaged exactly on the right infinity mark when the pentagonal prisms are placed in front of the rangefinder.

With different deviations of the pentagonal prisms, other marks may be chosen which unite stereoscopically at other distances, or which, on account of their difference of shape, could not possibly be united by stereoscopic vision.

2. *The König Adjuster.** This is a modification of the Abbe adjuster in which one of the pentagonal prisms (*e.g.* A) is rotated through an angle of 90°, the height of this prism being reduced by about one half. We will consider the case in which A has been rotated in the clockwise direction.

A beam of parallel rays incident in the direction of the arrow (Fig. 148) is split up into two portions. One portion, say the lower half, enters the left prism and is deviated through 90°. It then enters the entrance window on the left of the rangefinder, filling the lower half of the corresponding objective, the upper half of which must be covered up. The other portion of the beam of rays passes over the prism A, is deviated through 90° by the prism B, enters the rangefinder at R and fills the upper half of the right objective; the lower half of the latter is covered up.

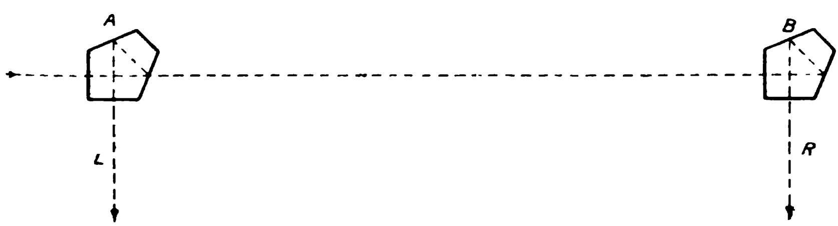


FIG. 148.

Since one half of each objective is covered up the illumination is naturally reduced, but this has no effect on the position of the image.

Limiting ourselves to the simplest case, suppose the deviations of the two prisms are equal; then the rays L and R are parallel, and the object under observation

* German Patent No. 165510; 1904.

which lies at a finite distance to the left, appears, with reference to the rangefinder, to lie at infinity. In this way the object serves to test the rangefinder at any time; when observing the object through the rangefinder, the latter must be set at infinity. Instead of using an object out in the landscape, we may replace it by means of a mark placed in the focal plane of a lens so that the mark is focussed sharply by the rangefinder.

3. *The Wild Adjuster.**—This arrangement depends on the following considerations: All recent rangefinders have pentagonal prisms at the extremities of their bases and if we assume that these prisms remain constant, the remaining part of the instrument, *i.e.* the double telescope with axes at 180°, is the only source of error. The most vital alteration that can take place in this double telescope is a displacement of the eye-piece prisms relative to the line joining the centres (more correctly the second principal points) of the two objectives in the plane of triangulation (*i.e.* the plane containing the base and the target). For example, consider the case of a coincidence rangefinder of the simplest kind with cross-reflectors 3 at the middle as in Fig. 149, the two rays L and R are shown in coincidence after reflection at these mirrors. The coincidence is considerably affected however, if either of the objectives 1 and 2, or the eye-piece reflectors, be displaced in the direction of the arrows.† Such a displacement is detected by Wild by

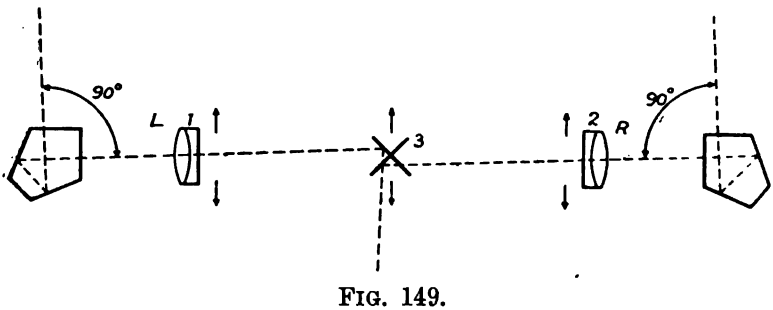


FIG. 149.

placing marks on the objectives 1 and 2 and observing them through a magnifying optical system placed in the position of the eye-piece. These marks are so placed, that when the parts 1, 2, and 3 are in their correct positions, the marks appear in coincidence, and any displacement of one of these parts is detected as a want of coincidence of the marks.

* German Patent No. 216420; 1907.

† In the figure, the centres of the two objectives and the intersection or "crossing" of the eye-piece reflectors all lie on a straight line.

B. Those included in the term "**The Absolute Adjuster.**"—In practice the angle of deviation of a pentagonal prism is not constant but is affected by changes of temperature and probably it changes quite considerably with time. The first public announcement of this fact was probably due to Barr and Stroud,* thus disturbing the supposition that had, up to that time, been the basis of the construction of rangefinders and adjusters. This fact became known to Zeiss through the measurements of Sieber. The question of avoiding this error has been very carefully investigated and described by Dennis Taylor, who proposes to return to the use of suitable angle reflectors in place of prisms; he improved these reflectors by using nickel steel possessing the same expansion as glass, and by particular methods of cementing.†

The adjuster arrangement of König described above, eliminates the error caused by an alteration of the deviation of the pentagonal prisms or reflectors, if a method introduced by Eppenstein be employed.‡

Employing this method, two parallel beams can be obtained with certainty. In the first place, two beams of parallel rays which are *diverging* relatively to one another are used, the angle between them being equal to the difference between the angles of deviation of the two pentagonal prisms; these two beams being made to enter the range-finder, a reading is taken on the latter. These two beams are made to *converge* relatively to one another, the angle between them being the same as before. Another reading is taken with the range-finder. In order to bring into coincidence in the instrument, first the diverging and then the converging beams, an optical element must be displaced; the amount of displacement is noted on a scale in the instrument. This scale is graduated in divisions that are proportional to the parallactic angles of objects at various distances. The mean of these two readings is taken, and the rangefinder corrected by that amount until it reads infinity. Thus the instrument is "set" absolutely; *i.e.*, independently of the variation of the deviations produced by the pentagonal prisms or reflectors.

The simplest construction of this type of adjuster is described below.

A beam of rays, approximately parallel, from a suitable distant object, or from a collimator (*i.e.*, a mark placed in

* English Patent No. 28728; 1906.
† English Patent Nos. 13562; 1907, and 20315; 1908.
‡ German Patent No. 221181; 1908.

the focal plane of an objective) is incident in the direction of the arrow from the *left*. One half of this beam, say, the lower half, is deviated by the left pentagonal and the upper half by the right pentagonal; the deviations produced by the two pentagonals may or may not be equal. The two partial beams then enter the rangefinder. In the upper part of Fig. 150 they are shown converging on one another.

A reading is taken on the two partial images produced in the field of view; for example, in a monocular rangefinder the two partial images are brought into coincidence by displacing the "deflecting prism" to the required extent. The rangefinder scale is attached to this deflecting prism and moves with it; a pointer is then set opposite the infinity mark on this scale. The position of the pointer is noted by means of a second scale, the adjuster scale.

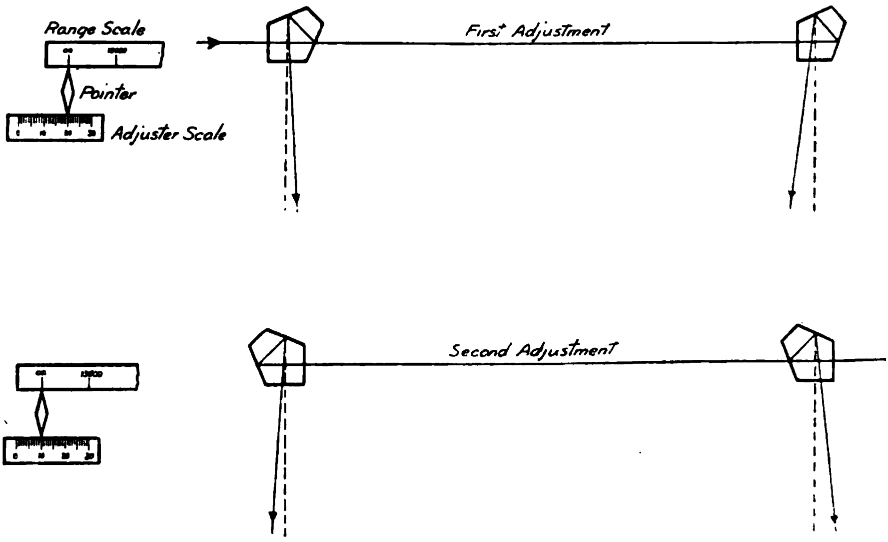


FIG. 150.

The two pentagonal prisms are then rotated through 90° in the plane of their principal sections; they then reflect into the rangefinder a beam incident from the *right*. Now, although incidental variations in the angle of deviation of a prism may be appreciable, the *rate* of variation is small, hence we may assume that the angles of deviation of the prisms remain constant during the two sets of readings. For the second set the procedure is just the same as above. The two partial beams, after leaving the pentagonal prisms, now *diverge* by an amount equal to their previous convergence. The two partial images are again brought into

coincidence by moving the deflecting prism and the pointer made to coincide with the infinity mark. The reading on the adjuster scale will now be different from the first reading. The mean of the two readings (in the figure the two readings are 10 and 20 and the mean 15) gives the position that the pointer would occupy if the two beams entering the rangefinder were exactly parallel, which would be the case if the pentagonals had exactly equal deviations, or if the light from a star had been used. If now the pointer be brought to the mean position on the adjuster scale, and the infinity mark on the rangefinder scale be made to coincide with it, then the rangefinder will be in correct adjustment and will give correct ranges.

VI. Construction of the Zeiss long base Rangefinder with Internal Adjuster.

The method of adjusting just described is independent of the angular errors of the deviating elements used, provided they do not alter between the two sets of readings; this is sufficiently approximate in practice, and any possible small change may be minimised by taking three, five or more sets of readings.

The internal arrangement of the rangefinder is illustrated in Figs. 151–153. Fig. 151 shows the paths of the rays during the taking of a range. Rays from the object under observation, after being deviated through 90° by the pentagonal prisms P_1 and P_2, are incident on the objectives O_1 and O_2, each of which produces an image of the object at the "separating prism" S. The two images are in this way separated by a horizontal line (*see* Plate I), and are observed through a terrestrial eye-piece consisting of the objective $\bar{O}$, prism P and the real eye-piece O_x. The step prism R is interposed to deviate the rays again, enabling the eye-piece to be placed in the same horizontal plane as the axis of the tube. The objective $\bar{O}$ is moved from the position shown in full lines to the dotted position when a change of magnification is required. The displacement of the partial images relative to one another is brought about by the prism K; this is placed in the right-hand end of the rangefinder, and consequently affects the rays forming the image produced by the right-hand objective O_2. The greater the distance of K from the image, the greater will be the effect of the deviation. A scale M is coupled directly to the prism K by means of the connecting

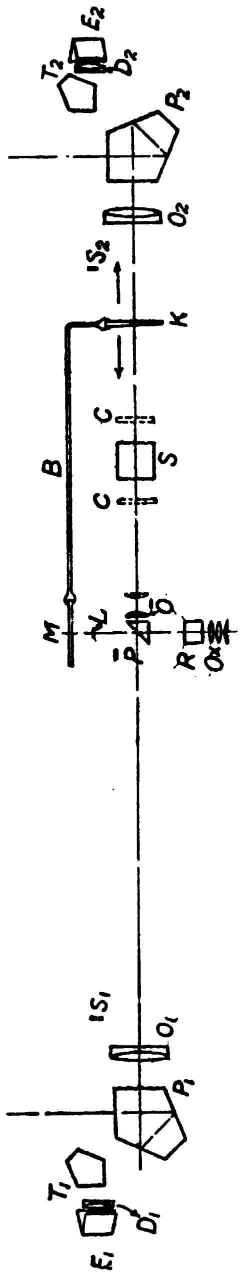
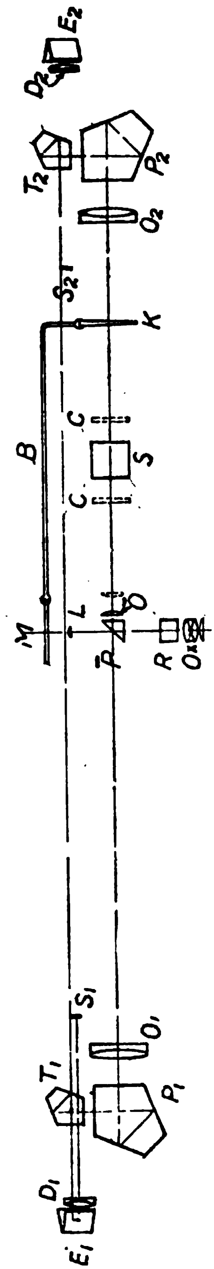


FIG. 151.



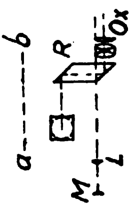


FIG. 152.

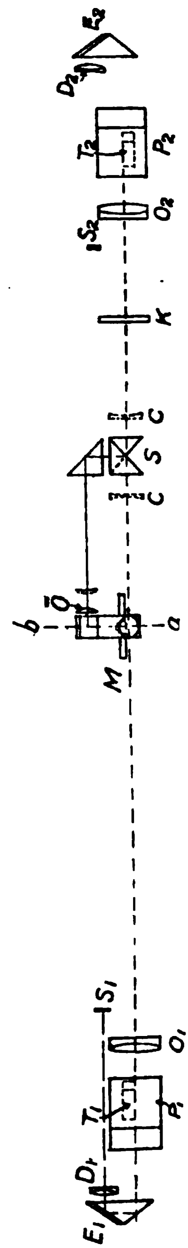


FIG. 153.

bar B. The ranges are engraved on the scale M. The latter is magnified by the lens L and viewed through the eye-piece O_x and the prism R; it appears at the top of the field of view and separated from it by a thin line. For taking ranges at night two cylindrical lenses or astigmatisers CC are provided, which may be inserted into the paths of the image forming rays; their effect is to make a point object (*e.g.*, a point of light) appear as a vertical line in the field.

Fig. 152 shows the paths of the rays during the process of adjusting. The adjuster prisms T_1 and T_2 are brought into position as shown in the figure, by turning a hand wheel until a feather is felt to snap into position. The mark on the strip S_1 is then seen in the field of view in the same way as any other object. Rays from the mark S_1 of the collimator D_1 are deviated through 180° by the right-angled prism E_1; the parallel beam is then deviated by the adjuster prisms T_1 and T_2 into the rangefinder and the first set of readings taken.

For the second set of readings the adjuster prisms are rotated through 90° and readings taken on the second mark S_2. This second position is not shown in the figures.

A vertical section through the rangefinder is shown in Fig. 153.

Fig. 154 shows an external view of a rangefinder of 3m. base.

The instrument is supported at the bearing rings AA which are 640 mm. apart; it rotates on rollers which are mounted on the cast-iron pedestal supporting the rangefinder. Thus by means of the handle G the instrument is easily moved in altitude and the object kept in the field of view. A body rest is provided on the pedestal, so that the motion in azimuth is the more easily controlled by the right hand which also moves the range working roller W. The instrument only requires unobstructed view in front of the objectives O, O.

Under ordinary circumstances the adjuster apertures at FF (between the eye-piece and right and left objectives respectively) and the scale aperture N (on the opposite side of the tube from the eye-piece), provide enough daylight illumination for the adjuster marks and range scale respectively. In order to render the adjuster marks and scale visible at night, or when the rangefinder is placed in an armoured turret, small electric lamps are provided, the current being supplied by an accumulator, or from a branch of the ordinary lighting

circuit. In the figure a lamp is shown at N only. The hand wheel T serves to move the adjuster in and out of position. It is rotated until a feather snaps into place, when the rangefinder is ready for the first adjustment. A further rotation as far as the stop provided, places the adjuster prisms into position for the second adjustment (*see* § 112, Sect. V, B on The Absolute Adjuster). The adjuster scale described in that section is engraved on the head J by means of which the pointer is moved.

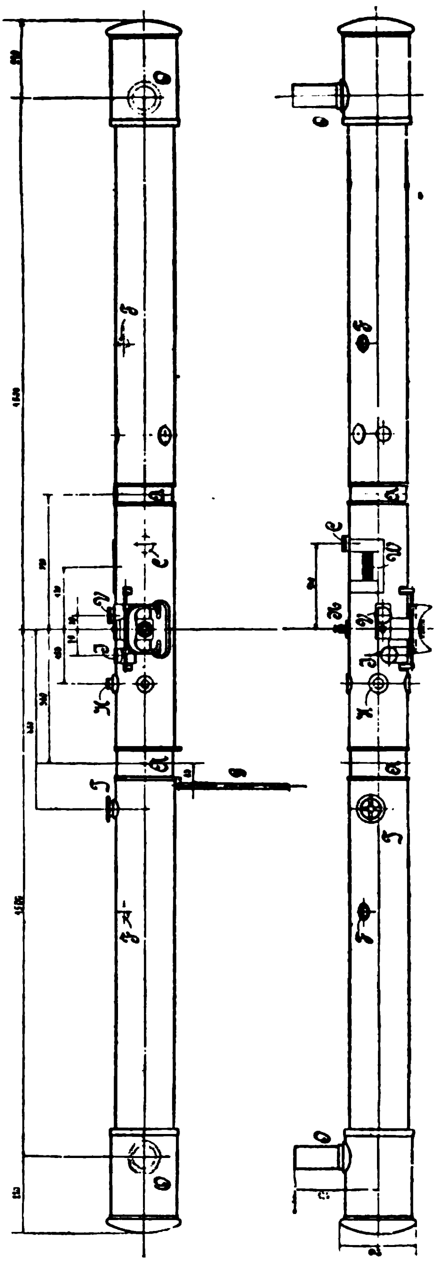


Fig. 154.

The astigmatisers are operated by the head C and the alteration of magnification by V.

113. The Goerz Rangefinder.

The rangefinder made by Goerz belongs to the group of monocular instruments, the principles of which have already been discussed in §§ 107-110. Both the coincidence and invert systems are used in this instrument. The measurement of a range is brought about by the displacement of a glass prism or a system of such prisms.

Previously the prisms in front of the eye-piece (the eye-piece prisms) were so arranged that their edges formed the separating line between the two partial images. These

prism edges were magnified by the eye-piece and appeared as a comparatively thick ragged line, which did not lend itself to accurate rangefinding. In order to produce as fine a separating line as possible, it has since been arranged that two prisms are cemented together with their surfaces of contact inclined to the axis of the eye-piece.

One part of the contact surface is silvered; and hence, of the light entering this prism system, part passes through unhindered whilst the remainder is reflected.

Fig. 155 represents a horizontal section through the rangefinder, that is, a section in a plane parallel to the plane of triangulation.

The front portion of the eye-piece prism system is composed of a rhomboid prism 9 and a triangular prism 10. This combination we will call the rhomboid prism. It serves to separate the two images from one another in the focal plane of the eye-piece. The paths of the beams of light are as follows:—

Considering the left-hand side of the instrument, the beam of light from the object is reflected by the pentagonal prism 4, transmitted through the objective 7, and reflected twice in the prism 12, its axis thereby being raised by an amount roughly equal to the diameter of the field of view. The beam is then incident on the roof prism 13 which deviates it through 90° and also causes an inversion of the image. Part of the beam now passes undeviated through the rhomboid prism and into the eye-piece, whilst the other part is stopped by the silvering on the cemented surface, thus taking no part in the image formation. The light entering the right-hand end of the instrument is reflected through 90° by the pentagonal prism 3, transmitted through the objective 6, and deviated again through 90° by the right angled prism 8; it is reflected at one face of prism 9. Part of the beam then passes through the cemented surface, taking no part in the formation of the image, whilst the other part is reflected at the silvered portion of the sloping face into the eye-piece. The silvered portion of the cemented face between prisms 9 and 10 is shown by means of a thick line in the figure, whilst the unsilvered portion is represented by a thin line. The edge of the silvered portion of the cemented face of the rhomboid prism 9-10 is the image separating line. Previously it had been usual to arrange the separating surface perpendicular to the plane of triangulation, with one side nearer to the eye-piece than the other. The result was that the separating line was

inclined to the eye-piece axis and consequently could not be focussed simultaneously throughout its entire length. In the arrangement just described, however, the cemented surface is inclined to the axis of the eye-piece so that its upper part is nearer to the eye piece than the lower. The edge of the silvered portion, *i.e.*, the separating line, lies throughout its length in the focal plane of the eye-piece. Since the prism 8 is not a roof prism, the upper image coming from the right, having suffered complete inversion in its passage through the objective 6, remains inverted.

The arrangement adopted for the measurement of the range is shown at 5 in Fig. 155. The two circular glass deflecting prisms are shown in the position in which they produce the maximum deviation of the beam. They are mounted in holders, round the periphery of which are fixed bevel wheels that are in gear with a driving bevel wheel placed between them. On the same spindle as this driving wheel is fixed a larger bevel wheel which gears with a third one fixed to one extremity of the range drum 14. On the outside of the range drum are spiral grooves in which the scale pointer moves as the drum rotates. Such an arrangement enables the scale graduations to be spread over a greater length than one single revolution of the drum; hence the graduations being further apart, become more widely separated than would be the case if the rate of revolution of the scale drum were the same as that of the deflecting prisms.

The determination of the range could be effected by one deflecting prism only. The necessary displacement of the image would be obtained by a rotation of the prism from one extreme position to the other; *e.g.*, from the position in which its thickest edge is nearer to the observer, to the position in which its thickest part is away from the observer. In the intermediate mean position the deflecting prism acts as a parallel sided glass plate in the plane of triangulation, but deflects the beam in a *vertical* plane; hence the upper image would be displaced in a vertical direction relative to the lower image by the rotation of such a deflecting prism. It is to prevent this relative motion of the images in a vertical direction, that two prisms are provided which rotate in opposite directions.

The optical elements of the rangefinder are arranged in various parts of the casing: the least sensitive parts, the pentagonal prisms and deflecting prisms, are supported in the outer casing 1. This latter consists of two rigid

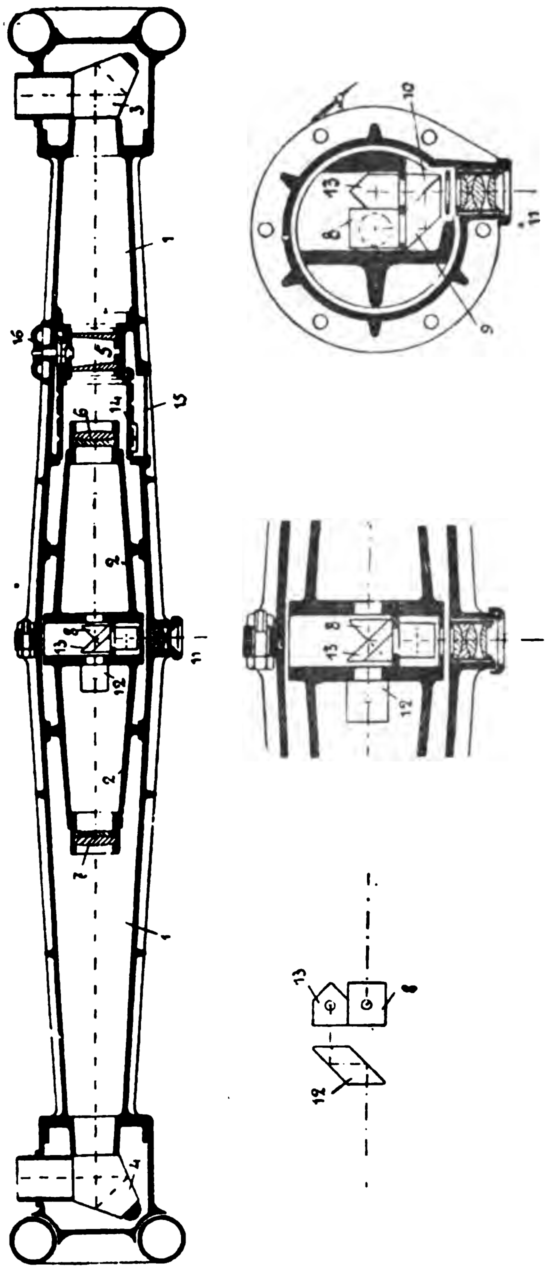


FIG. 155.

aluminium castings with longitudinal ribs, secured together in the middle by strong bolts. Those optical parts of the instrument for which it is more important that they should maintain their correct relative positions, *i.e.*, the two objectives 6 and 7 and the eye-piece prisms, are supported by a special inner tube 2. As shown in the drawing, this inner tube is supported by only two rings which lie in circular bearings within the outer casing, so that it is not affected by any possible distortions of the latter. Between the inner and outer casings there is accordingly a layer of air which serves to protect as much as possible the sensitive inner tube 2 from external temperature variations. This protected position of the sensitive elements of the rangefinder is important, since extremely small relative displacements of the objective and eye-piece prisms produce large errors in the determination of a range. As already mentioned the pentagonal prisms 3 and 4 are not absolutely insensitive to the effects of heat; their angles of deviation suffer small alterations in consequence of the strain which remains after annealing, even if the glass has been well annealed.

At the present moment, the firm of Goerz are engaged on the problem of replacing these pentagonal prisms by angle reflectors, *i.e.*, by parallel sided silvered glass plates which are mounted in a holder so as to include an angle of 45°. This holder is made of a material practically unaffected by temperature changes. Nickel steel has proved to be particularly suitable for this holder (*see* § 112 Section V, B).

114. Rangefinding by Means of Triple Mirrors.

If three planes SAB, SBC, SAC, pass through a point S (Fig. 156) a solid angle is formed ; when the inner surfaces of these planes are made reflecting, the arrangement is known as a **Triple Mirror.**

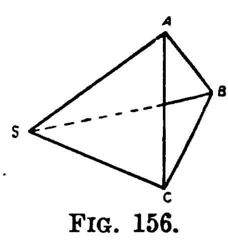


FIG. 156.

An incident beam of light is split up into several component beams by reflection at the three plane surfaces. The laws governing this reflection are dealt with in a work by Beck.* If the reflecting surfaces be perpendicular to one another, the arrangement is called a **Central Mirror.**† A characteristic property of a central mirror is that a beam of rays incident on it is reflected to the point of origin

* Zeitschrift für Instrumentenkunde, 1887 ; p. 380.
† See British Patent No. 21,856 1903, by Sir H. Grubb.

of the incident beam, quite independently of the orientation of the reflector. If a person look into a central mirror in a direction perpendicular to the plane ABC, an image of the eye is seen ; and this single image is continuously seen no matter how the reflector may be rotated (within a certain range of angle).

On account of their characteristic properties triple mirrors have been used for signalling purposes and for range-finding.*

The triple mirrors made by Zeiss are either central mirrors or very nearly so ; that is to say, the three reflecting planes may be exactly or almost perpendicular to one another. In order to obtain the maximum amount of illumination, the three reflectors are made from a single piece of glass ; the bounding plane at the front which is approximately equally inclined to the three reflecting planes, acts as entrance and exit face. Reflection inside the glass block takes place by total internal reflection. The optically inoperative parts of the block are cut off by three symmetrical planes.

In general a beam of rays incident on a triple mirror is split up into six partial beams which emerge from the mirror in different directions.

Signals may be sent from one place (the triple-mirror station or T-station) at which there is no source of light to

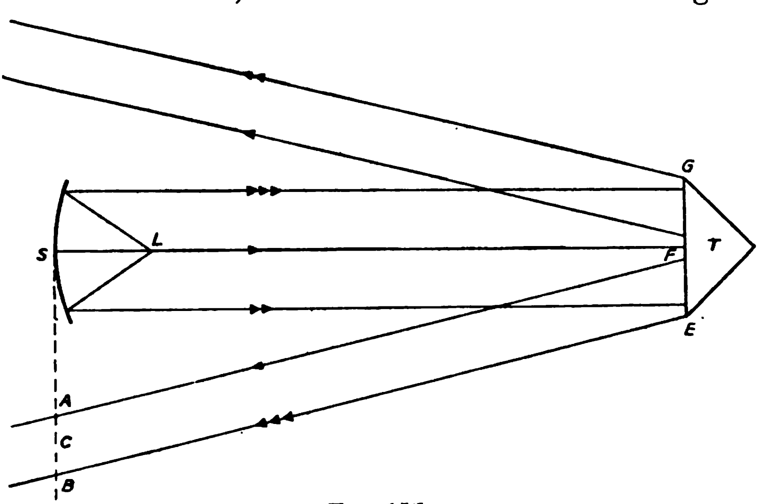


FIG. 157.

another station (the light station or L-station) at which the source of light is situated.

* See German Patent No. 187,418, by Zeiss of Jena.

In Fig. 157 let L be the source of light and S a searchlight reflector at the L-station ; the T-station, at which the mirror is erected, is some distance away, say 5 Km.

The beam of parallel rays with axis LF and incident on the mirror will be split up into two partial beams of parallel rays and reflected back to the L-station. If the mirror be erected in the correct manner, these two beams will arrive back at the L-station a little to either side of the original beam and may be observed at this station. If now the exit face GE of the mirror be alternately covered up and uncovered in some pre-arranged manner, signals may be sent from the T-station to the L-station.

As mentioned above, the three planes of the triple mirror may be perpendicular to one another ; or the angles between them may deviate slightly from right angles. This deviation may be so chosen that the distance SC bears a known small ratio to the distance SF, C being the centre of the cross section AB of the partial beam reflected back to the L-station. Usually it is arranged that

$$\frac{SC}{SF} = \frac{1}{10{,}000}$$

Hence if SC be measured and found to be, say $\frac{1}{2}$ m., then $SF = 5000 \text{ m} = 5 \text{ Km}$. In this way the triple-mirror serves for the approximate determination of ranges.

The superiority of the triple-mirror over other mirrors lies in the fact that within wide limits the paths of the beams are independent of the orientation of the mirror.

The triple-mirrors of Zeiss are made in two sizes—of 85 and 100 mm. diameter respectively. As a result of the particular method of construction of these mirrors only two partial beams emerge. The angles between the three plane mirrors differ by a small amount from right angles ; in consequence, the two emerging beams lie very close to and in the same plane as the incident beam.

115. Impression of Depth and Rangefinding with Monocular Observation.

Krusius has constructed an apparatus by which an impression of depth or relief is obtained by the use of one eye only.* This apparatus makes use of the physiological fact that an impression of the solidity or relief of an object can be obtained in monocular vision, if a succession of

* German Patent No. 221,067.

continually changing images of the object is presented to the eye; the differences in the images being such as would take place if the object were observed from different standpoints. The impression obtained is essentially the same as that which is usually called stereoscopic parallax, whilst the relief impression observed will differ from the ordinary binocular vision in that the object will not appear at rest but will appear in motion. Krusius also intended his method to be used for stereoscopic rangefinding with the employment of only one eye. The principle of the method is indicated in Fig. 158. The two objectives *r* and *l* are arranged at a definite distance apart, representing the base-length of the instrument. At *p* and *q* are two totally reflecting prisms; *u* and *v* are two right-angled eye-piece prisms placed one above the other. The latter are capable of being moved together in a vertical direction so that the beams of light from *r* and *l* may be reflected alternately into the eye-piece *x* and thence to the eye. As in the Zeiss stereoscopic rangefinder, glass diaphragms are inserted in the focal plane of the instrument; these diaphragms are provided with marks, corresponding pairs of which unite stereoscopically at different distances; by means of these marks the depth of objects in the landscape may be measured.

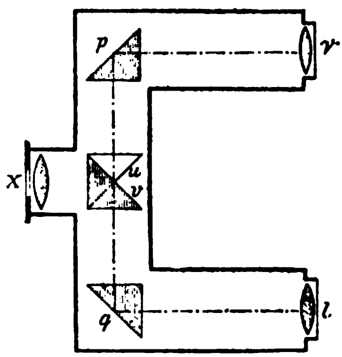


FIG. 158.

116. The Pulfrich Stereo-Comparator.

If the images of the landscape obtained by using a stereoscopic rangefinder be replaced by photographs with the same stereoscopic difference, and if these photographs be observed through a stereoscope provided with corresponding stereoscopic scales, we have the essentials of the *Pulfrich Stereo-Comparator*.

The stereo-comparator has already proved of importance in astronomy. For such purposes, very long bases, of the order of half the earth's radius may be chosen for the taking of stereoscopic photographs; in fact, if photographs of the stellar regions be taken at times corresponding to two diametrically opposite points of a diameter of the earth's orbit, this diameter is then the base. In this way a new

sphere of research has been opened up in astronomy. Previously bases of this kind had not been used for the taking of such photographs; generally changes in the positions of the heavenly bodies themselves were taken advantage of in order to obtain stereoscopically different images. In this way, stereoscopic images of the moon and of Jupiter with its satellites have been obtained. The stereo-comparator has also proved of great service in the sphere of photogrammetric measurement.*

117. The Pulfrich Stereometer.

The Stereometer accomplishes for near objects what the stereo-comparator does for distant objects.

The stereometer is so arranged that the negatives obtained when taking the photographs can be used directly without it being necessary to interchange them, as is the case in the usual stereoscopes. Thus by taking two stereoscopically different photographs of an external object, a permanent record is obtained, which may be submitted to measurement at any time, being thus of great value for many purposes. For example, the apparatus may be used in the following cases:—

- Skull measurements in anthropology.
- Identification of criminals.
- Artistic (sculptural) works.
- Measurements on bodies of living animals (important for breeding establishments and animal modellers).
- Research on zoological and medical preparations, and the study of the changes in tumours, &c.
- Archæological investigations.
- Measurements on historical objects of art.
- Determination of locality in criminal cases.
- Preparations of plans of excavations, &c.

* For further information on this interesting subject, see the work of Pulfrich "Neue Stereoskopische Methoden und Apparate."

Also the paper "Ueber die bis jetzt mit dem Komparator auf astronomischem Gebiete erhaltenen Versuchsergebnisse," in der astronomischen Wissenschaft; 37th year, Part 3.

CHAPTER XII.

Cystoscopes.

118. Positions of Entrance and Exit Pupils in Cystoscopes.

Cystoscopes are instruments used for the inspection of the human bladder. The instrument is introduced through the urethra, the bladder being filled with water (refractive index $n = \frac{4}{3}$). The lenses of the cystoscope are enclosed in a tube, which for the inspection of the human bladder has a length of about 200 to 250 mm. Usually the lenses should not exceed 5 to 6 mm. in diameter, and in certain cases they must be still smaller; thus the optical system of the instrument must conform to certain special conditions.

Suppose the optical system to consist of three systems, L_1, L_2, L_3; of which the first is the objective, the second the inverting system, and the third the eye-piece (Fig. 159). The object y under observation is situated at P, in water; an inverted image y' of this object is formed at P' by the objective L_1; the inverting system L_2 forms an image y'' of this at P'', the focal point of the eye-piece L_3. In the figure, the object y is shown at a distance of about 40 mm. from the objective, but this distance may vary.

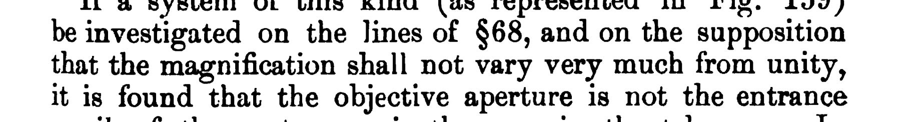


FIG. 159.

If a system of this kind (as represented in Fig. 159) be investigated on the lines of §68, and on the supposition that the magnification shall not vary very much from unity, it is found that the objective aperture is not the entrance pupil of the system as is the case in the telescope. In general, the entrance pupil is obtained by finding the image of the system L_2 formed in the fluid medium by the objective. Similarly the exit pupil will be obtained by finding the position of the image of the system L_2 formed by the eye-piece L_3. In Fig. 159, let R and R' be the centres of the entrance and exit pupils respectively. Let

$RP = p$; ρ and ρ' the radii of the pupils; and we will assume that the rays emerge from the eye-piece as a parallel beam. Then from equations (9) and (10) §72,

$$\frac{n\rho}{p} = D\rho' \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$v = Dd \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where D is the power of the whole system of the cystoscope, d the conventional distance of distinct vision, and v the magnification.

In Fig. 159, $\frac{\rho}{p} = u$ is the angle made with the axis by a ray leaving P and passing through the edge of the entrance pupil.

The product

$$nu = a$$

is called the **Numerical Aperture** of the system.

From equation (1),

$$a = D\rho'$$

Thus the numerical aperture of a cystoscope is equal to the power of the whole system multiplied by the radius of the exit pupil.

Important properties of the instrument, such as resolving power and illumination, depend on the numerical aperture.

From equation (2) we obtain

$$a = \frac{v\rho'}{d} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

As compared with that of a modern microscope, the numerical aperture of the cystoscope is very small; the optical requirements of such an instrument are, however, not very great, since it is more a matter of projecting the image than of obtaining a high magnification.

In the older form of cystoscopes of Nitze, the construction of which is similar to that represented in Fig. 159, the value of a lies between 0·002 and 0·004; in modern instruments it attains a value of 0·01, if the distance p of the object be assumed to be about 4 cm.

Assuming that a cystoscope should fulfil the condition that an object distant 250 mm. from the eye shall appear the same through the instrument as it does when viewed with

the unaided eye, then $v = 1$. If the length of the instrument from the eye-piece to the entrance pupil be, say, about 210 mm., the object distance p will be 40 mm., which is the usual distance when using a cystoscope.

From (1) and (2)

$$\frac{n\rho}{p} = \frac{v\rho'}{d} \quad \dots \quad \dots \quad \dots \quad (6)$$

We have $d = \frac{1}{4}$ m., $p = \frac{1}{25}$ m., $v = 1$, and $n = \frac{4}{3}$; hence

$$\frac{\rho'}{\rho} = \frac{25}{3} = 8\tfrac{1}{3}.$$

Thus on such assumptions, the ratio of the radii of the exit and entrance pupils is fixed. If nearer or more distant objects are observed with the instrument, then in order that an unaccommodated emmetropic eye may focus them sharply, the eye-piece must be displaced, or an additional lens used, thus altering the power of the whole system and the quotient $\frac{\rho'}{\rho}$. The change, however, is so small that it may be neglected with an instrument of this kind. Hence, if in equation (6) we put

$$\frac{\rho'}{o} = \textit{constant} = \frac{25}{3}, \text{ and } d = \frac{1}{4}\text{ m.}$$

then

$$v = n.\frac{\rho}{\rho'}.\frac{d}{p} = \frac{1}{25p} \quad \dots \quad \dots \quad (7)$$

where p is expressed in metres. Thus the magnification is inversely proportional to p; for nearer objects it increases appreciably. If $p = 4$ cm. $= \frac{1}{25}$ m., in which case the image is focussed most sharply, then of course $v = 1$. In the older instruments, the focal length of the objective was only a few millimetres; consequently objects only a few centimetres away as well as distant objects, were imaged in the immediate neighbourhood of the focal plane of the objective. Moreover the diameter of the emerging bundle of rays was usually less than 1 mm.; hence it is clear that for objects at different distances an instrument of this kind produced sharp images simultaneously in focus.

The images of distant objects produced by such an instrument are greatly diminished. This also follows from

equation (6). If an object at a distance of 3 m. be observed first with a cystoscope, and then with the unaided eye, we must refer the size of the object in the latter case, not to the conventional distance of $\frac{1}{4}$ m., but to the distance 3 m. In other words, in discussing the sizes of distant objects we must put $p = d$ (neglecting the length of the instrument). Also, since in this case the instrument is used in air, $n = 1$ and equation (6) gives

$$v = \frac{\rho}{\rho'}$$

Hence when the instrument is focussed for distant objects, the quotient $\frac{\rho}{\rho'}$, as is to be expected, actually gives the telescope magnification which in this case is less than unity. When $v = 1$ and the object distance is $\frac{1}{4}$ m. it was found above that $\frac{\rho'}{\rho} = 8\frac{1}{3}$, hence distant objects appear diminished $8\frac{1}{3}$ times.

For objects at the distance of distinct vision, the magnification of modern cystoscopes is usually low, being only about two-fold. If $p = \frac{1}{25}$ m. and $d = \frac{1}{4}$ m., equation (6) gives

$$\frac{\rho'}{\rho} = \frac{25}{3v}.$$

Thus the quotient $\frac{\rho'}{\rho}$ decreases as the magnification increases, and, when $v = 2$ takes the value $4\frac{1}{6}$.

119. Illumination of Cystoscopes.

Apart from the illumination of the object, which is effected by means of a small electric lamp, the brightness of the cystoscope image depends on the size of the exit pupil of the system, as in the case of vision through a telescope. In the old cystoscope systems of Nitze, consisting, as shown in Fig. 159, of three simple plano-convex lenses L_1, L_2 and L_3, the diameter of the exit pupil was at the most 1 mm. In the modern types, such as those made by Zeiss, and by Louis and H. Löwenstein, this diameter reaches 2 mm. or more, so that the illumination is increased four times or more.

A cystoscope of the type shown in Fig. 159 produces erect images. In order to obtain a suitable position for the source of light, a prism which deviates the rays through 90° is placed in front of the objective.

120. The Zeiss Cystoscope.

In the cystoscope made by Zeiss (designer M. v. Rohr),* the rays are first deviated through 90° by an Amici prism, the image thus undergoing complete inversion. The first image, formed by the objective, is consequently erect; it suffers two further inversions during the passage of the rays through the cystoscope, thus becoming finally erect. Fig. 160 indicates the paths of the principal rays. The two rays 1 and 2 from the object points O_1 and O_2 pass through the Amici prism and intersect at the small aperture in the diaphragm *b*; this aperture is the entrance pupil of the instrument. These rays again intersect at the lenses *f* and *h* which are to be considered as aperture diaphragms. Thus the image of the lens *f* formed by the system *e*, *d*, *c* is the entrance pupil, and the image of the lens *h* formed by the system *i*, *k* is the exit pupil. In the figure, the position and size of the exit pupil is shown at *l*. The path of the operative rays is as follows: an image of the object O_1 O_2 is formed by the objective at the position of the field-lenses *d*, *e*; this image is inverted by the lens *f* and projected to the lens *g*; the lens *h* again inverts it and projects it to the position *i*. The lens at *i* may be looked upon as a field-lens of a Ramsden eye-piece, the eye-lens being *k*.

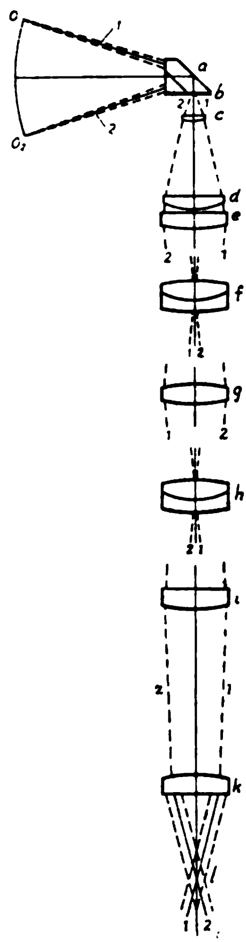


FIG. 160.

121. Cystoscope of Louis and H. Löwenstein.

The cystoscope made by Louis and H. Löwenstein, of Berlin, is represented in Fig. 161, in which the paths of the

* See American Patent No. 940,894.

principal rays are indicated; for example, the rays 1 and 2 from the object points O_1 and O_2. The lens L_3 is to be considered as the aperture diaphragm. The image of L_3 formed at R by the system L_2L_1 constitutes the entrance pupil, and the image of L_3 formed at R' by the system $L_4L_5L_6$, the exit pupil. A real inverted image of the object $O_1\,O_2$ is formed near the lens L_2 by the objective L_1 and the lens L_2, the latter acting essentially as a field-lens. The system $L_3\,L_4$ acts in principle as an inverting system forming a further image at L_5. L_5 is the field-lens and L_6 the eye-

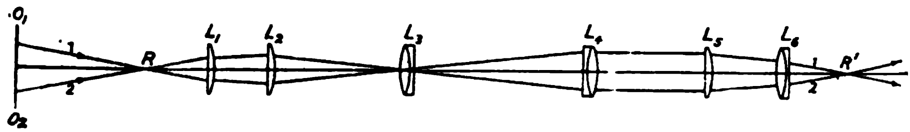


FIG. 161.

lens of a Ramsden eye-piece. If a totally-reflecting prism be inserted in front of the objective L_1, the incident rays are deviated through 90° but the image is reversed laterally. This reversion may be corrected by inserting an erecting prism (*see* §88) between the lenses L_3 and L_4, where the operative rays are parallel. The rays can, however, be deviated through 90° without lateral reversion of the image, if a pentagonal prism (§87) be arranged in front of the objective in place of the totally reflecting prism.

122. Other Cystoscopes.

The right-angled totally reflecting prism in front of the objective is avoided in the cystoscope of Wappler* by employing a silvered glass hemisphere which deviates the rays through 90° and acts at the same time as the objective.

In the "Endoscope" of Löwenstein† a second right-angled prism is placed in front of the usual objective prism; by rotating this additional prism, different portions of the interior of the cavity under observation can be viewed in succession.

Hirschmann‡ inserts a parallel-sided glass plate at an inclination of 45° with the optical axis between the eye-piece and inverting system. Part of the light from the object is thus deflected through 90° and may be used to

* German Patent No. 194,727 and American Patent Nos. 723,790 and 922,985.
† German Patent No. 177,780.
‡ German Patent No. 139,331.

photograph the object, whilst the observer can view the object by the remaining portion of the light which passes through the inclined plate.

"Endoscopes" in which illuminating tubes are provided alongside the observing tube have been described by Holb.*

In the stereoscopic cystoscope by Löwenstein† two optical systems are enclosed in tubes arranged close together, so that the combination can be introduced into the urethra. Totally-reflecting prisms are arranged at the eye-piece end of the instrument to deviate the two systems of rays twice through 90° so that the emerging rays are separated by a distance equal to the inter-ocular distance, and the instrument may be used in binocular vision.

APPENDIX.

In England the manufacture of Cystoscopes has only recently been undertaken. The cystoscope made by the Genito Urinary Manufacturing Company is illustrated in Fig. 161A.

The rays from the object O_1O_2 are received by the lens L_0 forming part of the objective and are then reflected at right angles towards the second objective lens L_0' by a roof prism. After passing the last named lens L_0', these rays are transmitted through a system of erecting lenses L_1, L_2 and L_4, L_5 to the Huygenian eye-piece L_6, L_7, whence they emerge parallel.

The image formed by the objective is inverted by the combination L_1, L_2, and is projected to a plane passing through L_3. After being inverted a second time by L_4, L_5 it is finally formed upright and right sided in the focal plane of the eye-piece L_6, L_7.

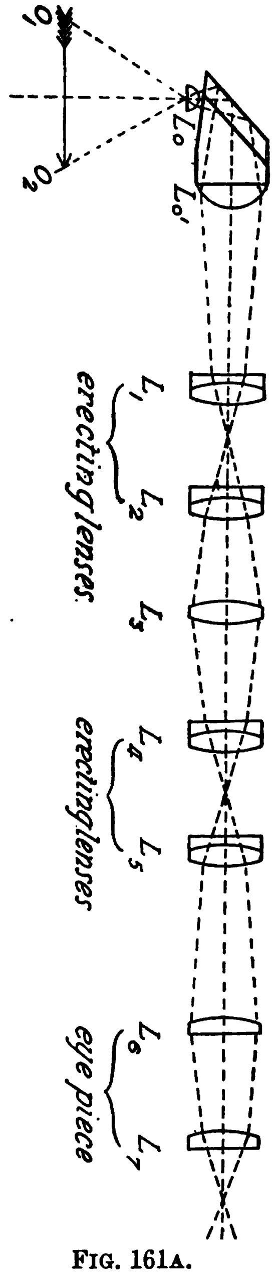


FIG. 161A.

The instrument has a length of 260 mm. The diameter of the lenses is 4·2 mm., and the magnifying power 1·8 to 2·3. The diameter of the field in air at the standard focal distance of 25 mm., is 26 mm.

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* American Patent Nos. 939,034 and 939,035.

† German Patent No. 164,966.

CHAPTER XIII.

The Microscope.

123. The Microscope. General Remarks.

Whilst the telescope serves for the observation of distant and relatively large objects, the function of the microscope is to present to the eye highly magnified images of near objects. The modern microscope consists of a very short focal length objective which forms a highly magnified image of an object, this object being situated a little beyond the focal plane of the objective system; the magnified image thus obtained then undergoes a further magnification by means of an eye-piece (usually of the Huygenian type). We are indebted to Abbe for the theory of microscopic imagery and for the high state of perfection reached in the performance of the optical system of this instrument, the most important part of which is undoubtedly the objective. Since the construction of the "apochromatic" objective systems by Abbe, there has been a pause in the development of this important, and from the point of view of design, most difficult optical system. It would appear as if the principles of both physical and geometrical optics, on which the Abbe theory is founded, are exhausted. Further progress may however be expected by the employment of good aplanatic systems with surfaces that are not spherical (*see* Chapter XVI on Aplanatism). Since in the largest majority of cases of microscopic observation, the object consists of extremely fine structures through which the light penetrates and is transmitted in the form of diffraction bundles, the illumination of the object plays a very important part. Concave mirrors had been in use for a long time for concentrating the light, until Abbe constructed the condenser named after him, which satisfies all the requirements of ordinary illumination by transmitted light ("bright ground illumination"). The discovery of ultra-microscopic observation by Siedentopf and Zsigmondy, and the development of illumination by means of oblique rays ("dark ground illumination"), as a result of which the resolving power is increased,* have given a new stimulus to the construction of condensers. Finally, mention must be made of the methods of photo-micrography which have been developed so successfully in recent times.

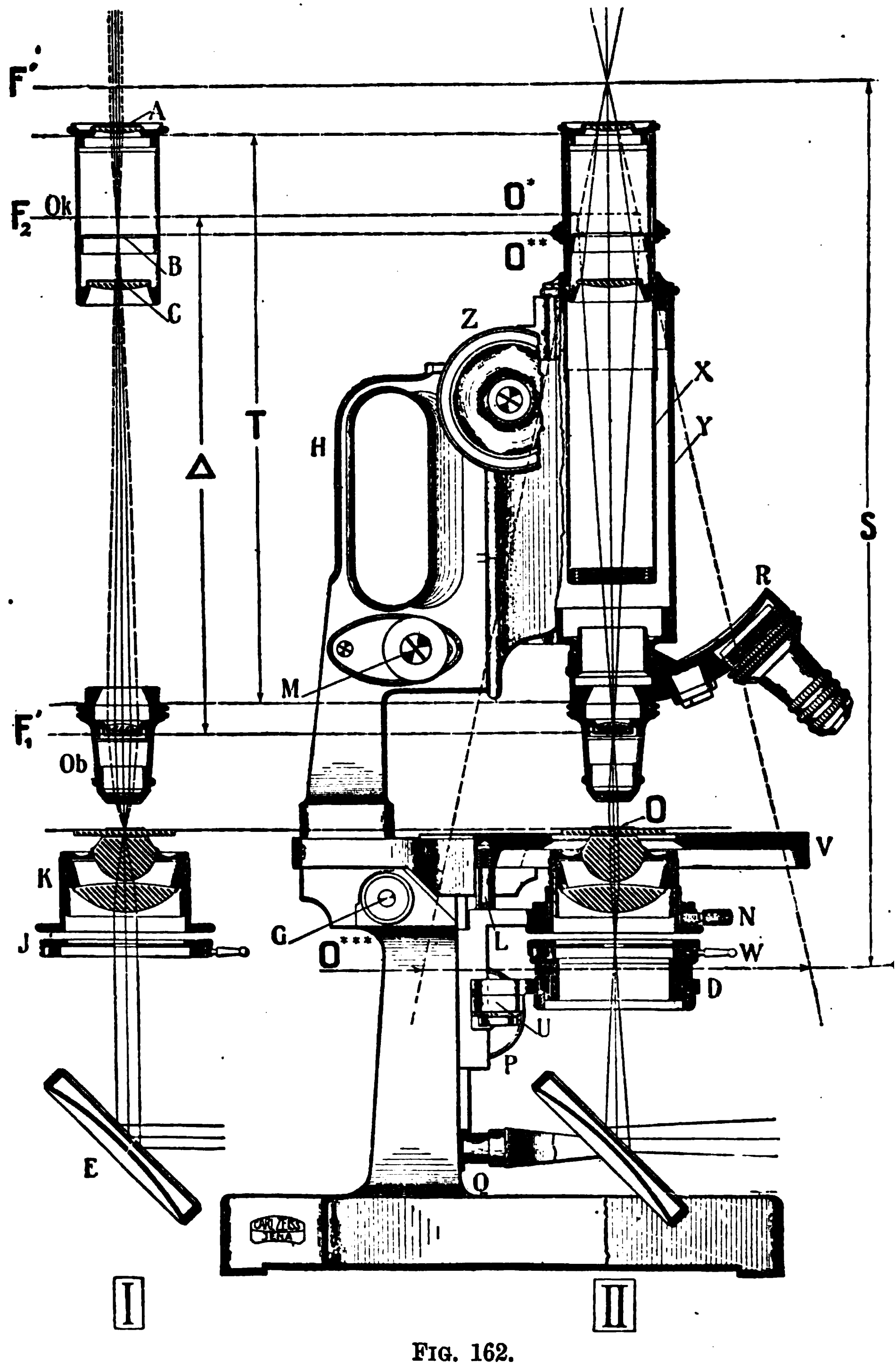
* This is doubtful, the more general view is that the resolving power is decreased.

124. Path of Rays in the Microscope.

We will consider a microscope of a comparatively low magnification made by Zeiss. Fig. 162, which is taken from a publication by that firm, shows clearly the various factors governing the image formation. The rays forming the image are shown in *I*, whilst *II* is a diagram showing the principal rays bounding the field of view. The objective represented in Fig. 162 is the one designated *AA* in the Zeiss catalogue; it consists of two compound systems. The distance between the anterior surface of the lower lens and the upper surface of the cover glass, that is, the *free working distance* or object distance, amounts to about 7·5 mm., the eye-piece employed being the one listed by Zeiss as the Huygenian No. 2. The equivalent focal length of the objective is 17 mm.; the first principal point of the objective, from which the focal length is reckoned, lies approximately midway between the component systems, whilst the anterior or lower focal point lies immediately above the object *O*. The posterior or upper focal point lies at the upper lens of the objective, as indicated in diagram *II* (Fig. 162) by the plane F_1', and will be called the upper focal plane of the objective.

Explanation of Fig. 162 (Drawn ½ full size.).

A Eye-lens of eye-piece.
B Eye-piece diaphragm (field-stop).
C Field-lens of eye-piece.
D Diaphragm carrier of Abbe illuminating apparatus.
E Illuminating mirror.
F' Posterior focal plane of whole microscope; *i.e.*, the plane perpendicular to the axis through the posterior focal point.
F_1' Posterior focal plane of objective.
F_2 Anterior focal plane of eye-piece.
G Hinge for inclining the microscope.
H Handle.
J Condenser system.
L Stop for limiting upward movement of illuminating apparatus.
M Micrometer head.
N Screw for clamping condenser to sliding sleeve.
O Object under observation.
O^* Real image formed in F_2-plane by objective alone.
O^{**} Real image of O^* formed by field-lens.


F'
A
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CARL ZEISS JENA
I
II

FIG. 162.

O^{***} Image formed by whole microscope in the plane of distinct vision.
Ob Objective (achromatic A.A.).
Ok Eye-piece (Huygenian No. 2).
P Focussing screw of illuminating apparatus.
Q Mirror carrier.
R Revolving nose-piece.
S Normal distance of distinct vision = 250 mm.
T Mechanical tube length = 160 mm.
U Axis of diaphragm carrier.
V Microscope stage.
W Button to open and close iris-diaphragm.
X Draw tube.
Y Outer tube.
Z Coarse focussing head.
Δ Optical tube length.

The anterior focal point of the Huygenian eye-piece lies between the component lenses as indicated by the plane F_2; below it is the eye-piece diaphragm B. The lower lens is the field-lens and the upper the eye-lens. If the field-lens were removed, the image O^* formed by the objective alone would lie in the F_2-plane. The image O^{**} formed by the objective and field-lens together lies in the plane of the eye-piece diaphragm.

This image O^{**} is observed through the eye-lens of the eye-piece. In Fig. 162 it is assumed that the eye of the observer is accommodated on or corrected for infinity; hence the eye-piece diaphragm lies in the anterior focal plane of the eye-lens. Rays from the various points of the real image O^{**} become parallel after refraction through the eye-lens; they consequently appear to come from the corresponding points of an image lying at an infinite distance in front of the microscope. This constitutes the virtual image of the object under consideration formed by the whole microscope.

In the diagram *II* are represented three principal rays from three separate points of the object; since they enter the objective parallel, they intersect above the objective at the posterior focal point of the latter, that is, in the plane F_1'; after refraction through the field lens the two outer rays pass through the edges of the eye-piece diaphragm. Hence the two points of the object from which these two rays originate appear at the edges of the field-of-view and consequently the eye-piece diaphragm constitutes the

field-stop. The angle included by these two rays on emerging from the eye-lens, is the angle under which the image formed by the whole microscope appears to the observer.

For an emmetropic eye at rest the image lies at infinity, but for the purpose of arriving at the magnification it is necessary to project this infinitely distant and infinitely large image on to a plane lying at the conventional distance of 250 mm. from the eye of the observer (this is the distance S in the figure). As has already been seen (§70) this projection plane is the position in which the object must be assumed to lie in order to obtain an expression for the magnification.

In diagram II. this plane is indicated by O^{***} (the image does not lie in this plane, it actually lies at infinity).

The ray shown traversing the microscope axis in diagram I. is the axis of the bundle of rays originating at the object point in the centre of the field of view ; it therefore represents the principal ray of the bundle. In a similar way, the two extreme rays in diagram II. are principal rays, being the axes of the bundles of rays emanating from the outermost part of the objective field-of-view and entering the objective.

In diagram I. only the bundle of rays from the axial point of the object is considered ; bundles of rays from other points of the object may be drawn in exactly the same way. If the course of the rays be followed closely, it will be seen that they are limited by the holder of the upper objective lens, and since this lens is filled with rays over the whole extent of its aperture, it acts as the effective aperture or "Iris" of the system.

The angle included between the axis and an extreme ray of the bundle originating from the object and entering the objective is called the **angular aperture.** Since the holder of the upper lens of the objective system determines the size of the angular aperture, it constitutes the **Aperture diaphragm.** It will be seen from diagram I. that the size of the aperture diaphragm is a measure of the aperture angles of the bundles of rays proceeding towards the image of the object. These bundles, previous to entering the eye-piece, are proceeding towards points of the image in the plane O^*, but owing to the converging effect of the field-lens, they are deviated so as to be directed towards the image lying in the plane O^{**} of the field-stop. They

emerge from the eye-lens as a parallel bundle and consequently form a sharp image on the retina of an eye accommodated on or corrected for infinity.

It is seen from diagram II. that the principal rays of all bundles originating at various points of the objective field-of-view intersect above the complete microscope at the posterior focal point of the latter in the plane F′. Further, the image of the aperture diaphragm of the objective formed by the eye-piece also lies in this plane—at least in the case considered. This image will be observed as a bright circle of light above the eye-piece and forms the **exit pupil** of the microscope. It is the common base of all bundles of rays proceeding from the microscope image. If the pupil of the observer's eye be brought into this plane, and the exit pupil of the microscope be not larger than the entrance pupil of the observer's eye, then all rays from the microscope image enter the eye.

The distance from the upper end of the draw-tube to the shoulder of the objective collar (represented in the diagram by T) is called the **mechanical tube length.** The objectives made by Zeiss are fixed in standard mounts and are adjusted for a mechanical tube length of 160 mm. Hence if an objective be screwed directly on to the tube, the distance between the upper and lower ends of the tube should be 160 mm. If on the other hand, the objective be fixed to some interchanging device, for example a revolving nose-piece or sliding objective carrier, account must be taken of the length of these fittings in reckoning the mechanical tube length. In the same way, allowance must be made if a special eyepiece, such as a drum micrometer eyepiece, a screw micrometer eyepiece, etc., be used, in which case the eyepiece lies some distance above the end of the draw-tube. The correct tube length must be adhered to rigorously whenever medium or high powers are being used; weaker achromats are not so sensitive to changes in tube length and consequently may be used with shorter or longer tubes without the definition of the image being noticeably impaired.

The distance from the posterior focal point of the objective to the anterior focal point of the eye-piece is called the **optical tube-length** or the **optical interval**; it is represented in the figure by the distance Δ. In the case of the objectives, called by Zeiss the "achromats," the optical tube-length assumes different values for the various combinations of objective and eye-piece, since the positions of the planes F_1' and F_2 vary considerably. On the other hand, Δ remains

almost constant for the different combinations of apochromats and compensating eye-pieces, being 180 mm. when the mechanical tube length is 160 mm.

The Abbe illuminating apparatus is shown in Fig. 162, the condenser consisting of a two-lens system of numerical aperture 1·20.

Comparing the rays indicated in the two diagrams of Fig. 162, it will be seen that in diagram I only those rays from the source of light are shown which intersect at the axial object point after reflection at the mirror and refraction through the condenser. Between the condenser and the mirror, and also between the mirror and the source of light (not shown in the figure), these rays are parallel. This is not, however, necessary. The illumination of the object may be effected by rays between the condenser and the mirror travelling in other directions; it is only necessary that all rays traced backwards from an object point through the condenser and the free aperture of the iris diaphragm, proceed in general to any one point of the luminous source. If this be the case, then conversely, light from this point of the source must proceed along the same path to the object point. It is quite immaterial whether the condenser forms a sharp image of the source of light or not in the object plane provided the extent of the source is sufficient. The illuminating bundles of light proceeding towards other points of the objective field-of-view may be indicated in precisely the same way.

In diagram II only the directions of those rays are indicated that intersect at the centre of the iris diaphragm after reflection at the mirror; in the case considered they are parallel after refraction through the condenser, the two outer rays passing through the edges of the field-of-view. Thus the aperture of the iris diaphragm acts as an infinitely distant source of light. The ray proceeding along the axis is the principal ray of the bundle shown in diagram I; the two extra-axial rays are the principal rays of bundles proceeding from points at the edges of the objective field-of-view; these are not shown in diagram I.

In diagram I the aperture angle of the bundle of rays emerging from the condenser is less than the aperture angle of the bundle of rays proceeding from the object point to the objective. Besides the rays which pass through the object without deviation—shown by full lines—there are also others, of which two are shown by dotted lines. These are the rays

which proceed to the periphery of the aperture diaphragm. In consequence of the structure of the object each one of the beams from the source of light suffers diffraction ; each bundle of rays in consequence is spread out into a wider bundle, of which, according to the aperture of the objective, a greater or smaller amount enters the microscope and takes part in the formation of the image (see §127). The dotted lines indicate the direction of two diffracted bundles which are able to enter the objective. Hence although the exit pupil is by no means filled with those rays of the illuminating cone of light passing *directly* through the object, the full aperture of the objective is made use of. Even if the illuminating bundle of rays had been assumed to be much narrower still, the diffraction bundles would spread out to such an extent in the case of an object of very fine structure that with central illumination the aperture of the objective would be completely utilised.

125. Power and Magnification of complete Microscope.

The equations (9) and (10), §72, derived for the general case of vision with the unaccommodated emmetropic eye, may be applied to the microscope. Thus calling the conventional distance of distinct vision S, as indicated in Fig. 162, the magnification is given by

$$v = D.S \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where D is the power of the whole microscope. Let the first and second focal lengths of the objective be f_1 and f_1', and of the eye-piece f_2 and f_2'. Since the eye-piece is situated in air, the quantities f_2 and f_2' are equal. If Δ be the optical tube-length, then

$$D = \frac{\Delta}{f_1' f_2} \ldots \quad \ldots \quad \ldots \quad (2).$$

From (1) and (2) putting $S = 250$ mm., the magnification becomes

$$v = \frac{250}{f_1'} \cdot \frac{\Delta}{f_2} \quad \ldots \quad \ldots \quad (3).$$

In the apochromatic systems of Zeiss, the magnitude $\frac{250}{f_1'}$, which is the magnifying power of the objective, is described as the **Objective Magnification,** whilst the magnitude $\frac{\Delta}{f_2}$, in the case of the compensating eye-pieces (which are used to eliminate the colour defects at the margins of the field), is known as the **Eye-piece number.**

For example, if an objective have a focal length $f_1' = 3$ mm., then the objective magnification is $\frac{250}{3} = 83$; and if the focal length f_2 of the eye-piece be 30 mm., then the eye-piece number is given by $\frac{180}{30} = 6$.

The **total magnification** or **magnifying power** of the complete microscope is obtained from the product of the objective magnification and the eye-piece number.*

126. Numerical Aperture of Microscope.

If the angular aperture, *i.e.*, the angle included between a given ray and the axis of the microscope, be u, then for paraxial rays

$$u = \frac{\rho}{p}$$

where ρ is the radius of the entrance pupil; in the case shown in Fig. 162, this is the radius of the second lens of the objective; as before, p is the distance of the object point from the entrance pupil.

Equation (9) §72 becomes

$$nu = D\rho'.$$

The quantity nu has already been called the numerical aperture. The rays entering the objective of a microscope cannot, however, be assumed to lie within the paraxial region, since the angle u may have values approaching 90°. Hence the angle u, supposing the system fulfils the sine condition, must be replaced by *sin u*. Thus, the numerical aperture in this case is given by

$$a = n \sin u = D\rho'.$$

* The magnification in the case of microscopic vision with an ametropic eye is derived from the author's "Einführung in die medizinische Optik," Leipzig, 1904 pp. 247–250.

NOTE.—The lower power achromatic objectives have been greatly improved in recent years in the matter of spherical and chromatic aberrations, the latter correction, however, being carried out for two colours only. Consequently these objectives produce images with a residual chromatic defect due to the secondary spectrum (*see* Chapter I.). A higher degree of achromatisation is introduced in the "Apochromatic Objectives," in which the secondary spectrum is eliminated, and three different colours are brought to a focus. Moreover, they are spherically corrected for two colours. Even then these objectives, if of large aperture, exhibit coloured fringes at the margins of the field arising from the chromatic difference of magnification; in order to correct this to a greater or less extent, a series of eye pieces was introduced, possessing an equal but opposite chromatic effect; they are called "Compensating Eye-pieces." The Huygenian eye-piece is generally used with the lower power achromats. "Orthoscopic Eye-pieces" furnish a higher eye-piece magnification than the Huygenian. *See* Czapski "Theorie der optischen Instrumente," Leipzig, 1904.—TRANS.

This equation also follows immediately from equation (4) §50, if a ray parallel to the axis and passing through the edge of the exit pupil be considered, thus interchanging the object and image spaces. In the case of *dry* objective systems, in which the medium between the cover glass and the lower lens of the objective is air, the numerical aperture a must be less than unity. If, on the other hand, the intervening space be filled with water (water-immersion objective) or oil (oil-immersion objective), then the values of a may reach 1·25 or 1·4 respectively. The essential properties of the microscope, such as illumination and resolving power, depend upon the numerical aperture.

127. Abbe's Theory of Microscopic Vision.

Fig. 162 indicates the paths of the rays of light through a microscope merely from the standpoint of geometrical optics. Actually, however, as was pointed out by Abbe, microscopic vision depends on the physical phenomenon of diffraction. It had been incorrectly supposed by earlier opticians that the resolving power of the microscope could be made continually higher by increasing the objective or eye-piece magnification. Abbe showed, on the contrary, that the resolving power is limited by the nature of the light itself and the laws of optical imagery. In Fig. 162 it is assumed that a real image of the object O is produced by the objective in the focal plane of the eye-piece.

According to the theory of Abbe, the formation of the image is largely determined by diffraction phenomena.* The light passing through the condenser and incident on the object is split up, in consequence of the fine structure of the latter, into diffraction bundles; accordingly there are produced diffraction patterns, consisting of various orders of diffraction spectra in the plane which, with reference to the objective, is conjugate to the source of light. If the source be a distant one, this plane will be the posterior focal plane of the objective. These spectra may be observed with the unaided eye by looking down the microscope tube when the eye-piece is removed. The finer the structure of the object, the more widely separated are these spectra. The diffraction bundles are, of course, united by refraction through the optical system of the whole microscope into a real image in the focal plane of the eye-piece, according to the

* For a clear and simple account of the part played by diffraction in optical image formation *see* the Cantor Lecture, "The Theory of the Microscope," by Conrad Beck, F.R.M.S.—Trans.

ordinary laws of image formation. Now it has been shown by Abbe that in order to obtain an image which shall be a reproduction of the object in all details, all the diffracted bundles, or at least those of appreciable intensity, produced by the object, must take part in the formation of the image; *i.e.* all the diffraction bundles must enter the objective. If the aperture of the microscope be too small, so that all the necessary diffraction bundles cannot enter the objective, then the image will be more or less dissimilar to the object. Generally, in order that details of a certain fineness may be resolved with central illumination, the light taking part in the image formation must include, in addition to the *direct* light, at least one diffraction beam on either side of the axis. The diffraction apparatus employed by Abbe enables this to be proved experimentally. On a plate coated with a thin film of silver is cut a series of fine lines; this serves as an object. With this apparatus the effect on the image formation of cutting out different diffraction beams may be demonstrated in a striking manner. If, for example, the diffraction beams of the first, third, fifth, &c., orders be cut out on either side, the image of the system of parallel lines is also a series of parallel lines but the lines are only half as far apart as in the object; the image thus appears doubly as fine as the object. That the magnification of a microscope is limited also follows from Abbe's theory.

Thus if

a be the numerical aperture of the microscope objective, and a_c that of the condenser, then from the theory of diffraction

$$\delta = \frac{\lambda}{a + a_c}$$

where λ is the wave-length of the light and δ is the smallest distance apart of the lines in the object that can be resolved. The **resolving power** of the microscope is inversely proportional to δ. If the light used to illuminate the object be axial, then $a_c = o$ and

$$\delta = \frac{\lambda}{a}$$

If the aperture of the condenser be equal to that of the microscope we obtain

$$\delta = \frac{\lambda}{2a}$$

which gives the maximum resolving power that may be obtained.

If, for example, we put $a = 1{\cdot}4$, an aperture which is realised in oil-immersion systems, then

$$\delta = \frac{\lambda}{2{\cdot}8}$$

Thus it appears from this that objects may be resolved which are separated by about one-third the wave-length of the illuminating light.

128. Dark Ground Illumination.

As we saw in the discussion on the Abbe theory of microscopic vision, the resolving power is determined by the value of δ. This quantity δ, *i.e.*, the distance between two elements of the object, becomes smaller the larger the numerical apertures of the microscope and condenser. To obtain the maximum effect, the former must not be less than the latter as otherwise all the rays of the condenser do not enter the objective, and thus can have no effect upon the resolving power.

Increasing the numerical aperture of the condenser has the effect however of emphasising the *contrast*. If by some suitable means the peripheral rays of light from the condenser which enter the objective direct are stopped out, then the usual microscope image, which is dark on a light background, becomes what is called **a dark-ground image**; that is, the object shows up *brightly* on a *dark* background. With dark-ground illumination the effect obtained is not an increase of resolving power but a production of greater contrast. It is possible by this method to make many objects quite distinctly visible which would be invisible or difficult to see with ordinary light-ground illumination. Several recent forms of condenser for producing dark-ground illumination are described below.*

129. The Heimstädt Reflecting Condenser.

A reflecting condenser was described by Stevenson in 1879; more recently a condenser of this type has been devised by Heimstädt.† This latter condenser makes it possible to use either light-ground or dark-ground illumination. The arrangement is such that a lens and a diaphragm are mounted on a sliding plate or on a wheel diaphragm, either of which may be caused to move past the reflecting lens *s*. In addition the disc or plate is provided with an

* A very accurate description of the development of the condenser is given in :—"Die Vorgeschichte der Spiegel-kondensoren" by H. Siedentopf in "Zeitschrift für wissenchaftliche Mikroskopie," Vol. 24, 1908, pp. 382–395.

† German Patent No. 217,229.

aperture g so that ordinary direct illumination may be used. In Figs. 163 and 164, s is the reflecting lens, l the lens in front

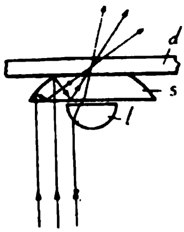

FIG. 163.

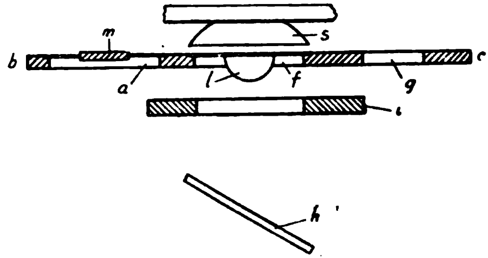

FIG. 164.

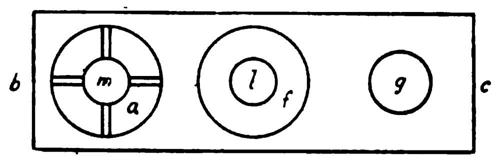

FIG. 165.

of it and d the object stage, connected with the reflecting lens by immersion. The paths of the rays are indicated by arrows. The metal plate $b\ c$ (Figs. 164 and 165) is capable of lateral movement, so that the apertures a, f and g may take up their positions in succession in front of the reflecting lens s. If, for example, the plate be moved to the right, then the aperture a (provided with a stop m) comes into position, and the apparatus serves for dark-ground illumination ; if the plate be made to take up the position shown in Fig. 164 so that the aperture f, with the lens l, comes into position in front of s, we obtain the apparatus for illumination by transmitted light. The mirror h is the ordinary microscope mirror.

130. The Zeiss Paraboloid Condenser.

A reflecting parabolic condenser was devised by Wenham in 1856 but was not successful. Later, this idea was improved and a form of parabolic condenser was put on the market by Zeiss. Fig. 166 represents the apparatus.

P is a plano-convex block of glass, the curved faces forming a paraboloid of revolution. B is the central stop which cuts out rays of aperture 0 to 1·1. The focus of the paraboloid lies at the upper surface of the object stage, and between the paraboloid and the object stage is the layer I of immersion liquid.

As in the simple methods of stopping out employed in the immersion condenser, the diffraction fringes in the image disappear in the case of the paraboloid condenser, in

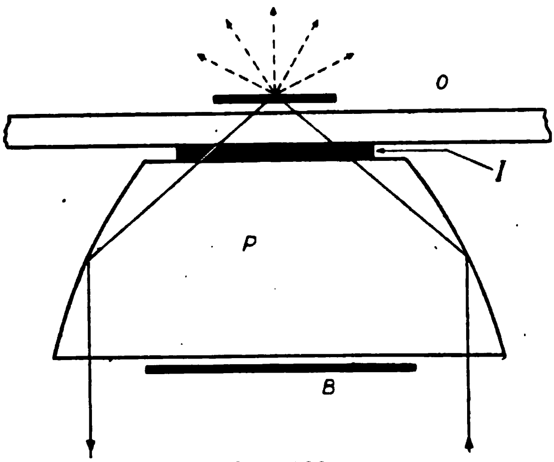

FIG. 166.

consequence of the annular form of oblique illumination; the fringes only appear with eccentric illumination when isolated portions of the illuminating ring are cut out.

Since, with reflecting condensers, refraction is avoided, this type of condenser is free from chromatic aberration. There is freedom from spherical aberration because of the well-known property of the parabola that incident rays parallel to the axis are united exactly at a single point.

The illuminating rays have numerical aperture 1·1 to 1·4. Dark-ground illumination depends on the fact that these rays are totally reflected at the upper surface of the cover-glass when the medium lying above the latter is air. Thus in Fig. 166 the ray incident on the right hand side returns on the left; the light diffracted by the object is shown by dotted lines.

131. The Siedentopf Cardioid Condenser.

The image formation resulting from the combination of a spherical reflecting surface and a cardioid surface is aplanatic; this will be seen later in the chapter on aplanatism. By this is understood that not only is spherical aberration eliminated, as in the paraboloid condenser, but in addition the sine condition is fulfilled. Approximately aplanatic image formation ensues if, as in the case of the cardioid condenser, the surface be not a true cardioid but be replaced to some extent by a spherical surface.

This condenser is shown diagrammatically in Fig. 167. It is made in two parts which are cemented together, the surface of separation *a* being spherical. The size of the annular diaphragm *b* is so arranged that the numerical aperture of the rays passing through amounts to about 1·1 to 1·3. The shaded portions are made of glass. The incidence rays are first reflected outwards at the spherical surface *c*, then at the outer concave spherical surface *d* and become united at the object. The narrow space between the object stage *e* and the condenser is filled with immersion liquid.

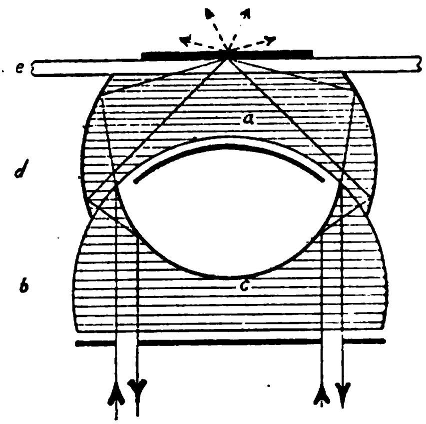

FIG. 167.

132. The Ignatovsky Reflecting Condenser.

A form of reflecting condenser devised by Ignatovsky and placed on the market by Leitz of Wetzlar resembles the cardioid condenser in outward appearance.

In Fig. 168 the outer reflecting surface *aba'd'* and the inner one *cdb'c'* are both spherical. The two portions I and II are combined with the spherical surface *ecc'e'* between them. The incidence rays ABA'B', unite after the various reflections, in the focus P, and emerge in the directions PA, PB., PA' and PB'. If the medium above the cover glass be air, then total reflection takes place at the upper surface of the glass. The numerical apertures of the extreme rays PA and PB lie between 1·0 and 1·4.

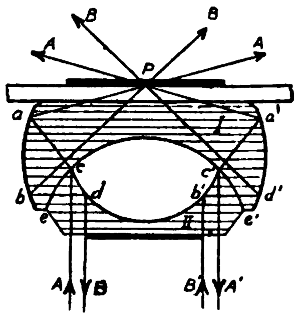

FIG. 168.

133. Photo-micrography with Ultra-violet Light.

According to the equation $\delta = \frac{\lambda}{2a}$, the resolving power of a microscope may be increased by increasing the numerical aperture or by decreasing the wave-length of the light. If

the former method be used a limit is soon reached. Thus the numerical aperture $a = n \sin u$ may be increased by increasing the refractive index of the immersion liquid. With the refracting media available the maximum value for the aperture is about 1·4, and hence a limit is reached in the amount by which we can increase the resolving power on that account. A further increase in resolving power, however, may be obtained by using light of the shortest practicable wave-length. This has been done with success by Köhler*, in his methods of photo-micrography.

In photo-micrography, the image produced by the objective is not used directly, since it would be too small; the whole microscope itself is used as a projection system. For this purpose, the eye-piece is displaced to some extent so that the emerging bundles of rays become slightly convergent, uniting to a real image on the screen. In order to obtain the same impression when viewing this projected image as is obtained when looking through the microscope in the ordinary way, the camera extension of the projecting apparatus must be made equal to the distance of distinct vision of the observer, *i.e.*, about 25 cms. Zeiss modify the eye-piece by replacing the eye-lens by a system which is spherically and chromatically corrected. The image is thus very good for definition and further is in sharp focus for any position of the projection screen. For the purposes of photo-micrography, the light used by Köhler is of a very short wave-length, being obtained by means of a spark discharge between cadmium and magnesium electrodes, and light of longer wave-length is eliminated. Since the eye cannot see the light of these short wave-lengths, such light cannot be used for subjective observation. The eye is most sensitive to the green light of wave-length 550 $\mu\mu$, whilst the wave-length of the ultra-violet light is only in the neighbourhood of 275 $\mu\mu$.† Again, ordinary glass absorbs the ultra-violet rays to a great extent, and consequently the whole optical system of the microscope used in such a photo-micrographic apparatus must be made of a material which is

* See the paper "Mikrophotographische Untersuchungen mit ultraviolettem Licht" by Dr. A. Köhler (Zeitschr. für wiss. Mikrosk., Vol. 21, 1904, pp. 129–165 and 273–304).

† Since the wave-length of light is so small, special units are used in its measurement—

1 micron $= 1\mu = 10^{-3}$ mm.

1 micro-millimetre $= 1\mu\mu = 10^{-6}$ mm.

1 Ångström unit $= 10^{-7}$ mm. $= 0{\cdot}1\,\mu\mu$ (this is equal to 10^{-10} metres hence it is sometimes called a Tenth-metre).—Trans.

transparent to these rays. The only materials suitable are Quartz and Rock Crystal. The crystalline structure of the latter must be reduced by melting, otherwise double refraction would take place. Unmelted Rock Crystal may be used with success only for weaker eye-pieces. Since in photography with ultra-violet rays, the light used is monochromatic, it is not necessary to correct the microscope objective for chromatism.

134. The Zeiss Micro-Projection System.

The following particulars refer to a double objective suitable as a micro-projection system* ; it is corrected for spherical and chromatic aberration and coma over a large aperture-ratio.

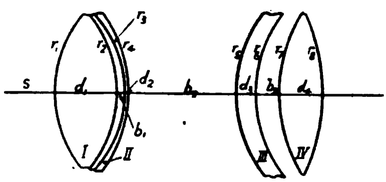

FIG. 169.

Constructional Data.

Focal Length	...	... ...	27·2
Aperture Ratio	...	... ...	1 : 1·8
Numerical Aperture		... ...	0·28

Radii.	*Thicknesses and Separations.*
$r_1 = 48·0$	$s = 15·6$
$r_2 = -20·59$	$d_1 = 3·5$
$r_3 = -11·53$	$b_1 = 1·8$ (air separation)
$r_4 = -15·98$	$d_2 = 1·0$
$r_5 = 101·01$	$b_2 = 7·4$ (air separation)
$r_6 = 25·5$	$d_3 = 1·0$
$r_7 = 35·9$	$b_3 = 0·8$ (air separation)
$r_8 = -17·15$	$d_4 = 3·5$

Glass.

I and IV.	II and III.
$n_C = 1·51339$	$n_C = 1·70863$
$n_F = 1·52144$	$n_F = 1·73280$

* German Patent No. 186,473 by Carl Zeiss of Jena.

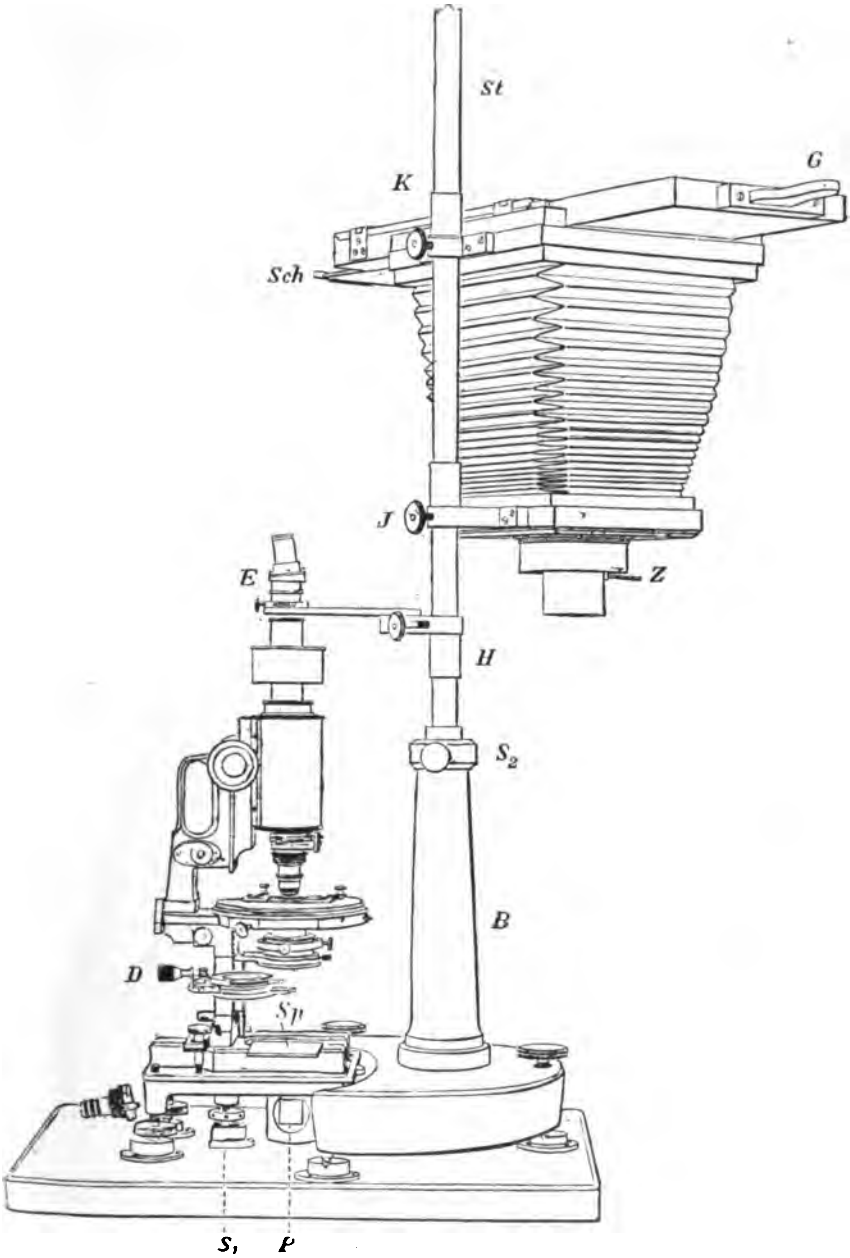

FIG. 170.

S_1	Screw for fixing base plate of microscope.
P	Reflecting prism of Quartz, which reflects the horizontal incident light into the microscope.
Sp	Plane mirror for observing the spark image on the uranium glass plate.
D	Diaphragm carrier with uranium glass plate; shown displaced to one side.
B	Base of camera support.
S_2	Screw for clamping rotatable pillar *St.*
H	Movable carrier for finder *E*.
J K	Movable carriers for camera.
Z	Time shutter.
Sch	Shutter of dark-slide.
G	Handle of frame holding photographic plate.

A form of photo-micrographic apparatus made by Zeiss is illustrated in Fig. 170; with the microscope and camera in the positions shown, the apparatus is ready for finding and focussing. Above the eye-piece is the finder *E*, and the camera is rotated to one side around the pillar *St.* In order to take a photograph the finder *E* is rotated to one side and the camera with its lower connecting piece brought into position over the tube of the microscope.

FIG. 171.

In Fig. 171 is shown a photograph taken with ultra-violet light ($\lambda = 275\ \mu\mu$) of the diatom *Amphipleura pellucida*, which is often used as a test object for a microscope; magnification 3,600 diameters.

135. Ultra-Microscopy.

The discovery of ultra-microscopy is due to Siedentopf and Zsigmondy. By this term is understood the rendering visible for observation, particles which are smaller than the smallest that are resolvable according to the Abbe theory of microscopic vision. These objects are called microns.

Micrones are in general distinguished, as Submicrones and Amicrones according as they can be made visible or not. So far as our present knowledge goes, the limit between these two classes appears to be about $4\mu\mu$ which is therefore in the neighbourhood of the dimensions of a hydrogen molecule, the diameter of which is calculated to be $0{\cdot}1\mu\mu$. The question of rendering visible ultra-microscopic particles differs from the image-formation of fine structures, in that it comes about by means of diffracted rays distributed equally in all directions, the micrones thus acting as self-luminous bodies. If these diffracted rays proceeding outwards in all directions be collected by a lens, then an image point is formed, which, in consequence of the effects of interference. is surrounded by coloured rings. Siedentopf has enumerated certain conditions that must be fulfilled in ultra-microscopic image-formation, viz.:

1. The illumination must be very brilliant.
2. There must be a great contrast between the illumination of the small diffraction pattern and the background.
3. The illuminating apparatus must possess a large numerical aperture.
4. The illuminated layer must not be appreciably thicker than the depth of focus of the objective.
5. The distance between the particles must be resolvable, *i.e.* they must not be concentrated to too great an extent.*

Two different forms of ultra-microscopic apparatus are made by Zeiss, *viz.*: the Slit Ultra-microscope and the Cardioid Ultra-microscope. The first is adapted for the investigation of solid bodies; the second for fluids.

The Slit Ultra-microscope of Siedentopf and Zsigmondy is illustrated in Fig. 172 in which

a is the supporting table.
b „ optical bench.
d „ arc lamp diaphragm.

* *See* Fischer "Ueber Ferienkurse für wissenschaftliche Mikroskopie" Zeitschrift für wiss. Mikroskopie; vol. 27, part I., p. 110.

f is the projection lens.
g ,, fine adjustment head for slit.
h ,, second projection lens.
i ., microscope stand.
k ,, optical base-plate.
l ,, cross-wire.
m ,, micrometer screw for effecting the displacement in the horizontal plane.

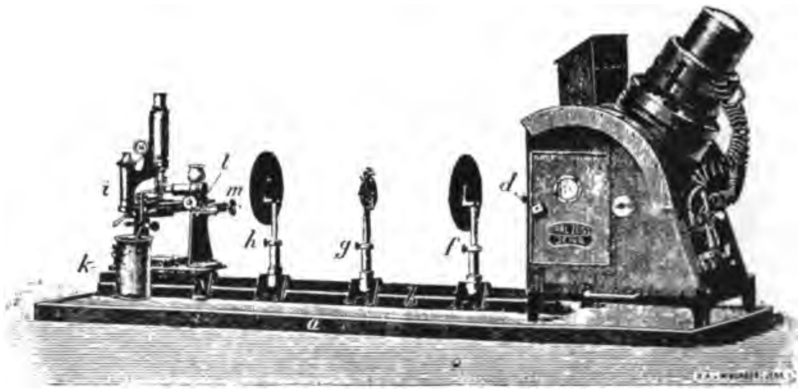

FIG. 172.

Fig. 173 illustrates the Cardioid Ultra-microscope.

a is the supporting table.
b ,, optical bench.
d ,, arc lamp diaphragm.
e ,, screen with convex lens.
f ,, vessel containing water.
g ,, microscope with Cardioid condenser.
h ,, base-plate.

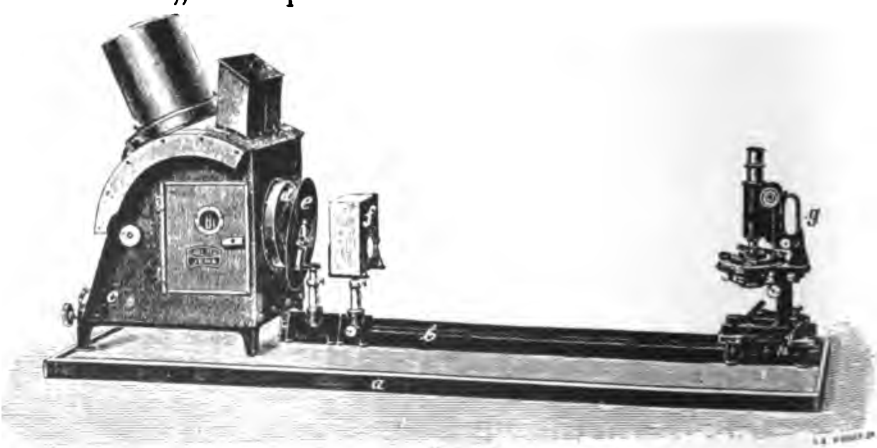

FIG. 173.

Ultra-microscopy has already been extensively applied. It is well adapted for the demonstration of the Brownian movement of molecules, to the investigation of the gold ruby ·glasses, and to the observation of a great number of micro-chemical reactions.

The most important literature on this subject is given below. It is outside the scope of this book to enter into the details of researches that have been carried out in this subject.

N. GAIDUKOV. Dunkelfeldbeleuchtung und Ultramikroskopie in der Biologie und Medizin. Jena, 1910. Published by Gustav Fischer.

R. ZSIGMONDY. Zur Erkenntnis der Kolloide. Jena, 1905. Published by Gustav Fischer.

—— Colloids and the Ultramicroscope, a manual of colloidal chemistry and ultramicroscopy. Trans. by Jer. Alexander. New York, 1909. John Wiley & Sons.

A. COTTON and H. MOUTON. Les ultramicroscopes; les objets ultramicroscopiques. Paris, 1906. Published by Masson.

SIEDENTOPF. Ueber die physikalischen Prinzipien der Sichtbarmachung ultramikroskopischer Teilchen. Berl. klin. Wochenschr. No. 32, p. 7, 1903.

—— Dunkelfeldbeleuchtung und Ultramikroskopie. Zeitschr. wiss. Mikr. 24, 13-20, 1907.

—— Ueber mikroskopische Beobachtungen bei Dunkelfeldbeleuchtung. Zeitschr. wiss. Mikr. 25, 273–282, 1908.

—— Die Sichtbarmachung von Kanten im mikroskopischen Bilde. Zeitschr. wiss. Mikr. 25, 424–431, 1908.

—— Ueber ultramikroskopische Abbildung. Zeitschr. wiss. Mikr. 26, 391-410, 1909.

—— Ueber einen neuen Fortschritt in der Ultramikroskopie German Phys. Soc., . 12, 6–47, 1910.

—— Ultramikroskopische Untersuchungen über Steinsalzfärbungen. German Phys. Soc. 7, 268–286, 1905, and Physikal. Zeitschr. 6, 855–866, 1905.

—— Ueber kolloidale Alkalimetalle. Zeitschr. Elektrochem., 12, 635-637, 1906.

—— Ueber künstlichen Dichroismus von blauem Steinsalz. German Phys. Soc. 9, 621-623, 1907, and Physikal. Zeitschr. 8, 850-852, 1907.

—— Ueber ultramikroskopische Abbildung (Prelim. Communication). German Phys. Soc. 11, 574–576, 1909, and Physik. Zeitschr. 10, 778–780, 1909.

—— Lichtreaktionen im Kardioid-Ultramikroskop. Zeitschr. für Chemie und Industrie der Kolloide 6, 3–6, 1910.

—— On the rendering visible of ultramicroscopic particles and of ultra-microscopic bacteria. Journal. Roy. Micr. Soc. 573–578, 1903.

—— Ueber die Umwandlung des Phosphors im Kardioid-Ultramikroskop. Report of German Chem. Soc. 43, 692–694, 1910.

CHAPTER XIV.

The Photographic Objective.

136. General Remarks.

Photographic objectives consist essentially of converging systems and furnish real images of objects. When in use, the sensitised plate is placed in the image plane of the objective. A great deal of progress has been made during the last few years in the development of the optical performance of photographic systems; they are now successfully corrected for spherical and chromatic aberrations and astigmatism and are made to fulfil the sine condition ; in addition, complete flattening of the image has been accomplished. The most important types of objectives, together with graphical diagrams of their corrections, are very well described in the work of von Rohr, "Theorie und Geschichte des photographischen Objectives," 1899. The constructional data of the types developed during the last decade or so will be given later in the chapter.

If ρ is the radius of the entrance pupil of an objective and ϕ its focal length, then $\frac{1}{\phi} = D$ is the power. The quantity

$$\frac{2\rho}{\phi} = 2\rho D = M \qquad \ldots \qquad \ldots \qquad (1)$$

is called the **Aperture Ratio** or the relative aperture of the objective. This quantity M is important, as we shall see below, in that it determines the illumination, on which the 'rapidity' of the lens depends. The magnification, *i.e.* the size of the image compared with the size of the object, may be calculated from the formulæ given in §§ 33–35. Let x be the distance of the object y from the first focal point of the objective ; then the size of the image is given by

$$y' = y \cdot \frac{\phi}{x}$$

For this object to be imaged sharply, the photographic plate must be placed at a distance x' from the second focal point, where

$$x' = \frac{\phi^2}{x}$$

The effective diaphragm of the objective may be placed in front of or behind it, or may be placed between its component parts; it is, however, seldom placed behind. If the diaphragm be placed in front of the lenses (on the same side as the object) then it is at the same time the entrance pupil; if, however, the diaphragm be in any other position, the entrance pupil is determined, as we have already seen, by finding the image of the diaphragm formed by the portion of the system lying in front of it.

The magnification ratio of the pupils, *i.e.* the ratio of the radii of the entrance and exit pupils, represented by

$$B = \frac{\rho'}{\rho}$$

is an important quantity in photographic optics. In symmetrical objectives,* the diaphragm is usually placed midway between the component parts; in this case the pupils lie in the principal planes and therefore $B = 1$. Further, in the case of most unsymmetrical systems, B is approximately equal to unity; it is only in the case of tele-objectives, to be described later, that the quantity B assumes appreciably smaller values. In photographic objectives, the errors dealt with in Chapter V., namely spherical aberration, astigmatism, and distortion, are eliminated and the sine condition fulfilled.

137. Depth of Focus of Photographic Objectives.

The questions of blurred-image formation and depth of focus have already been discussed in §§41 and 42, and in §66 a numerical example on the depth of focus of the eye was given. The rules there enunciated may, of course, be applied directly to the image formation by photographic systems.†

Fig. 174 will help to illustrate the matter. R and R' are the centres of the entrance and exit pupils. Two pairs of conjugate planes with axial points P, P' and P_1, P_1' are shown; P is conjugate to P' and P_1 to P_1'. Two bundles of rays filling the pupils are indicated; for the sake of clearness one is shown emanating from the axial point P_1 while the

* Compound photographic systems usually consist of two component systems I. and II.; when the two component systems I. and II. are identical but oppositely placed, the compound system is said to be symmetrical; usually the stop is placed midway between the two components. From equations X. and XI. § 33, it follows that the pupils lie in the principal planes of the compound system.—TRANS.

† It is the depth of focus in the object space that is here considered. The term "depth of field" is often applied to represent this quantity by British photographic writers; it will sometimes be referred to here as "object-space depth."—TRANS.

other originates at the extra-axial point Q_1 in the P_1-plane. The latter bundle converges in the image space on the extra-axial image point Q_1' in the P_1'-plane. If the screen be placed in the P'-plane, then all object points in the P-plane will be imaged sharply on it. Object points in any other plane of the object space (*e.g.* Q_1 in the P_1-plane) will not be imaged sharply, and in place of a sharp image *point* there will be produced a small *blur-circle* of centre m and diameter ae — for the present, astigmatism and other aberrations of the image will be neglected.

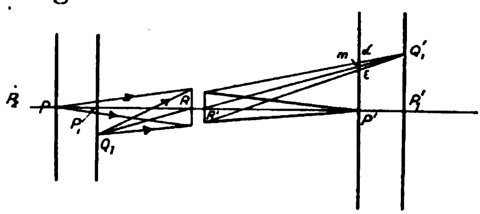

FIG. 174.

In this way a blurred image of the object P_1Q_1 is produced. The size of this image will be $P'm$ or $P'a$. Both the sizes of the blur-circle and the blurred image may be obtained from the equations (9) and (10) §41.

If we conceive the object space to be filled with objects at varying distances from R, then images more or less blurred will be formed on the screen. This imagery of space in the P'-plane will be called a *complete image-formation.* Sharp images will be produced of objects lying in the P-plane. The images of points, lying in planes in the immediate neighbourhood of this plane, will consist of blur-circles so small that the eye may not be able to distinguish them from points. Whether the eye distinguishes them as points or circles depends, firstly on the resolving power of the eye (see §109), and secondly on the distance of the eye from the screen, Let l represent this latter distance and z_0 the radius of the largest blur-circle that cannot be differentiated from a point at that distance, then

$$z_0 = \frac{w_0 \, l}{2} \dots \quad \dots \quad \dots \quad (1a)$$

and equation (11) §42 becomes

$$Q_1 = Q \pm \frac{w_0 \, l}{2 \, \rho'}(\Delta - Q) \quad \dots \quad (2)$$

The positive sign gives the *near* and the negative sign the *far* object-space depth.

Q_1 is the complete vergence corresponding to the extreme positions in which the object may be placed and still appear to be sharply imaged. (see §§39-41).

The last equation may be modified to suit the case of a symmetrical objective, in which $\rho = \rho'$.

From §40

$$Q = \frac{\rho^2}{p} \qquad Q_1 = \frac{\rho^2}{p_1} \qquad \Delta = D \cdot \rho^2$$

hence

$$\frac{1}{p_1} = \frac{1}{p} \pm \frac{w_o l}{2\rho} \cdot \left(D - \frac{1}{p}\right) \qquad \ldots \qquad (3)$$

Ex. 103. Let $D = 10$ *Dp.*, $p = 1$ m., $l = \frac{1}{4}$ m. and $\rho = 0{\cdot}01$ m. If the resolving power of the eye be assumed to be one minute of arc, what are the near and far object-space depths of focus?

From equation (3)

$$\frac{1}{p_1} = 1 \pm \frac{0{\cdot}00029}{4 \times 2 \times 0{\cdot}01} (10 - 1)$$

$$= 1 \pm \frac{1}{30}$$

$$\therefore p_1 = 1\frac{1}{29} \text{ m. or } \frac{30}{31} \text{ m.}$$

Thus all objects that are further from the entrance pupil of the objective than $\frac{30}{31}$ m. and nearer than $1\frac{1}{29}$ m. will appear distinctly on the screen.

138. Depth of Focus in the case of subsequent Enlargement of the Photograph.

If a photograph be enlarged m-fold, either by projection or subsequent photographic processes, we have the case of optical image formation of an object (viz. the photograph) which lies wholly in a plane. Then since the blur-circles of the original photograph become m-times larger in the subsequent enlargement, the object-space depth is diminished for the taking of the original photograph.

Thus in equation ($1a$), z_0 must be replaced by mz_0 and we obtain for the radius of the largest permissible blur-circle in the first photograph,

$$z_o = \frac{w_o l.}{2\, m} \qquad \ldots \qquad \ldots \qquad (3a)$$

and the general equation for the object-space depth becomes

$$Q_1 = Q \pm \frac{w_o l.}{2\, m\, \rho'} (\Delta - Q)$$

or for the case of symmetrical objectives,

$$\frac{1}{p_1} = \frac{1}{p} \pm \frac{w_o l.}{2\,m\,\rho}\left(D - \frac{1}{p}\right) \quad \dots \quad (4).$$

Ex. 104. A photograph of the landscape ($p = \infty$) is to be enlarged 100 times by projection and observed from a distance of 10 m. What is the object-space depth if $\rho = 0{\cdot}01$ m., $D = 10\,Dp$, and $w_o = 0{\cdot}0003$?

Since we are dealing with an infinitely distant object, we are only concerned with the near depth. Assuming the objective to be symmetrical, equation (4) gives

$$\frac{1}{p_1} = \frac{w_o l}{2\,m\rho}\,D = \frac{0{\cdot}0003 \times 10 \times 10.}{2 \times 100 \times 0{\cdot}01} = 0{\cdot}015$$

or

$$p_1 = \frac{1}{0{\cdot}015} = 66{\cdot}6 \text{ m.}$$

Thus we see imaged sharply in the final enlargement all objects which were distant more than 66·6 m. from the camera when the original photograph was taken. This depth is rather small. If ρ be halved and D doubled, then p_1 becomes 16 m.

139. New Expression for the Depth of Focus.

Equation (2) § 137 may be modified in a striking manner. Thus,

$$\frac{1}{p_1} - \frac{1}{p} = \frac{z_o}{\rho'}\left(D\frac{\rho'}{\rho} - \frac{1}{p}\right)$$

in which z_o may be positive or negative.

For the case under consideration, equations (3) and (4) § 39 take the forms

$$\frac{\rho}{\rho'}\cdot\frac{1}{p} + \frac{\rho'}{o}\cdot\frac{1}{p'} = D$$

and

$$\beta = \frac{p'}{p}\cdot\frac{\rho}{\rho'}$$

Eliminating ρ' and D we obtain

$$\left.\begin{array}{l} \dfrac{1}{p_1} - \dfrac{1}{p} = -\dfrac{z_o}{\rho}\cdot\dfrac{1}{\beta}\cdot\dfrac{1}{p} \\ \text{for the far object-space depth, and} \\ \dfrac{1}{p_1} - \dfrac{1}{p} = \dfrac{z_o}{\rho}\cdot\dfrac{1}{\beta}\cdot\dfrac{1}{p}. \end{array}\right\} \quad \dots \quad \dots \quad (5)$$

or the near o bject-space depth.

Thus, if a certain value for z_0 be agreed upon and the objective aperture and magnification are given, then for a given distance p of the object, the object-space depth is fixed. **The type of objective and its focal length are quite immaterial, having no effect on the object-space depth**; thus the depth for a tele-objective (*see* § 146) is the same as for any other objective whose entrance pupil coincides with that of the tele-objective. On the other hand, in the case of near objects, the depth can be modified by a displacement of the entrance pupil of the system.

For instance, if $\frac{z_0}{\rho} = \frac{1}{100}$ and $\beta = 1$, then the near depth is given by

$$\frac{1}{p_1} - \frac{1}{p} = \frac{1}{100\,p}$$

or

$$p_1 = p \cdot \frac{100}{101}.$$

Thus if the object be distant 101 cm. from the entrance pupil ($p = 101$), then $p_1 = 100$ cm. and the near depth is 1 cm. If, without changing the quantities z_0, ρ and β, the entrance pupil be moved another 101 cm. away from the object, (by altering the position of the aperture diaphragm), then $p = 202$ cm. and the depth, $p_1 - p = 2$ cm., *i.e.*, double the previous value.

Consider an eye situated at R, the entrance pupil of the photographic objective, and suppose it is focussed on the object P; its refraction is then $P = \frac{1}{p}$. If now the eye observe another object point P_1 (Fig. 174) it assumes another refraction condition $P_1 = \frac{1}{p_1}$ by virtue of its power of accommodation. The quantity

$$P_1 - P = \frac{1}{p_1} - \frac{1}{p}$$

is in ophthalmic optics known as the **Power of Accommodation** of the eye (*see* § 61). In the present connection we will call it the **Near Depth Power** T_n. Similarly we obtain as the far depth power (T_f) the accommodation which the eye must exert in order to focus sharply from P_1 to P. In each case this quantity may be expressed as equivalent to the effect of a convex lens of power $P - P$ placed in front of the

eye, or it may also be measured directly by the change of refraction of the crystalline lens of the human eye (*see* § 64). It will thus be seen that the depth power is a quantity whose dimensions are those of a power and is measured in dioptres.

Hence

$$T_n = T_f = \frac{z_o}{\rho} \cdot \frac{1}{\beta} \cdot P \ldots \qquad \ldots \qquad (6)$$

where P is the refraction of the eye placed in the position of the entrance pupil of the objective and focussed on the object to be photographed. For the total amplitude of the depth we have:

$$T = T_n + T = 2\frac{z_o}{\rho} \cdot \frac{1}{\beta} \cdot P.$$

140. Illumination of Projection Systems and Photographic Objectives.

The rules governing the illumination of optical instruments for visual purposes are dependent upon the laws of illumination of projection systems, to which class the photographic objective belongs. The brightness of the image of an extended object depends of course upon the illumination of the object; it also depends on the distance of the object from the objective and on the losses due to reflection and absorption within the latter. We will now consider only the dependence of the illumination on the type of objective and on the distance away of the luminous surface. The luminous surface will be assumed to lie perpendicular to the optical axis, and at a distance x from the first focal point of the objective. If the distance of this focal point from the centre of the entrance pupil of the instrument be p_o, then the illumination is given by the expression :*

$$H = c \cdot \frac{\pi \rho^2}{f^2} \cdot \left(\frac{x}{x + p_o}\right)^2$$

where c is a constant depending essentially on the illumination of the object; f is the focal length of the objective, and ρ the radius of the entrance pupil. Referring to equation (3), Chap. IV., and putting $p' = \infty$, we obtain for the value of p,

$$p_o = f \cdot \frac{\rho}{\rho'} = \frac{f}{B}$$

* A proof of this expression with reference to the system of the eye is given in Gleichen "Einführung in die medizinische Optik." Leipzig, 1904, p. 141 *et seq.* The result is of general application, however.

where B is the magnification at the centres of the entrance and exit pupils.

Introducing the aperture ratio $M = \frac{2\rho}{f}$, then

$$H = \frac{\pi}{4} c \, . \, M^2 \, . \left(\frac{1}{1 + \frac{f}{x} \cdot \frac{1}{B}} \right)^2 \quad \ldots \quad (7)$$

As before $\frac{f}{x} = \beta$ is the lateral magnification at the object and image points.

For very distant objects, as in the case of photographs of landscapes, $x = \infty$; hence

$$H = \frac{\pi}{4} . \, c \, . \, M^2$$

or, for very distant objects the illumination is proportional to the square of the aperture ratio.

In symmetrical objectives, in which the aperture diaphragm lies at the mid-point, *i.e.* $B = 1$, we have

$$H = \frac{\pi}{4} c \, . \, M^2 \left(\frac{1}{1 + \beta} \right)^2 \ldots \quad \ldots \quad (8)$$

As the magnification β increases, *e.g.*, in photo-micrography, H decreases very rapidly. When the image and object are equal in size, *i.e.*, $\beta = 1$, the illumination decreases to one-fourth of the illumination obtained in photographing distant objects.

In the case of unsymmetrical objectives, for given values of M and β, the illumination may be enhanced by making B as large as possible. If B be made infinitely great, the exit pupil moves to infinity whilst the principal rays become parallel to the axis in the image space, then the illumination is constant for objects at all distances, its value being

$$\frac{\pi}{4} c \, . \, M$$

If, as is often the case, the illumination decreases towards the edges of the photographic plate or projection screen, this diminution is proportional to the fourth power of the cosine of the angle of inclination of the rays in the object space.*

It should be remarked that these rules do not apply to objects which, by virtue of their great distance for example,

* **For a proof of this statement** ***see*** **Gleichen "Lehrbuch der geometrischen Optik." Leipzig, 1902, p. 400** ***et seq.***

appear as points, such as the fixed stars. In this case the illumination is directly proportional to the objective aperture. For photographs of all terrestrial objects, however, the above equations for illumination are applicable.

141. "True-to-Nature" Photographic Image-Formation.

The consideration of blurred image-formation and depth of photographic systems leads up to the question of the formation of images which will appear true to nature. This problem may be illustrated as follows :

The eye, S_1 of the photographer looking out into space, fixes the particular object that he wishes to photograph. A sharp image of this object will be formed on the retina, whilst all other objects, in so far as they are visible, will be imaged on the retina with varying degrees of sharpness. Image-formation of this kind, we have already called "complete" image-formation. Let us suppose that the camera objective is focussed on the "fixed" object and a photograph taken. The latter may then be enlarged say m-times either by projection or by re-photographing the original photograph. The enlarged image may then be observed at a certain distance by an eye S_2, this eye receiving a retinal image of the "fixed" object and of all the other visible objects. This latter retinal image will only be identical with the "complete" image formed on the retina of the eye S_1, if the image-formation is "true-to-nature." Then the photograph appears to the eye S_2 exactly as the actual landscape appeared to the eye S_1. The problem then, is to construct an objective which will produce images true-to-nature. It is not at once certain that an objective possessing this property exists, since very many definite conditions must be fulfilled ; thus *all* object-planes must be imaged on the image-plane in such a way that there is perfect agreement (some planes sharp, others more or less indistinct) in the two cases of observation with the eyes S_1 and S_2.

The author has discussed mathematically the problem of this kind of image-formation in his paper "Die Grundgesetze der naturgetreuen photographischen Abbildung," Halle a. S., pub. by W. Knapp, 1910.

From this paper, two conditions which must be satisfied by the objective are :

1. The entrance pupil of the objective must be equal to the entrance pupil of the eye

2. The power D of the objective is determined by the equation

$$D = \frac{P}{B} + mL \quad \dots \quad \dots \quad (9)$$

in which P and L are the refractions of the eyes S_1 and S respectively, referred to their focal points (§ 55), and B is the lateral magnification at the entrance and exit pupils of the objective.

With regard to the first condition above, § 65 shows that the size of the entrance pupil of the eye differs very little from the size of the pupil, so that for ordinary purposes we may neglect any difference between them. Since the diameter of the pupil varies between 2 mm. and 8 mm., the diameter of the entrance pupil of the objective should be chosen within these limits.

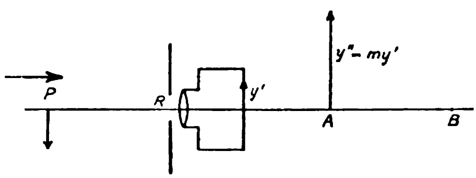

FIG. 175.

In order to investigate the second condition, Fig. 175 represents a camera on the plate of which is produced an image y' of an object y situated at P. Suppose that a second image of size $y'' = my'$ be produced at A by some subsequent magnifying process.

Consider first the eye S_1 which observes the object y, its first focal point being situated at the centre R of the entrance pupil of the objective ; let the reciprocal value of the distance PR measured in metres be P (not to be confused with the point P). when P is the refraction of the eye S_1.

At B is situated the first focal point of the second eye S_2 which observes the photograph. The reciprocal of the distance AB expressed in metres is the refraction L of the eye S_2.

Whilst a rigid proof of equation (9), which is one of the conditions for true-to-nature image-formation must be omitted here, it will be shown that this equation is not in disagreement with the known laws of optical image formation ; also that it does, at any rate, state the condition that the eyes S_1 and S_2 receive equal retinal images of the *sharply imaged*

object y. It does not, however, necessarily follow that all other planes which are not sharply imaged will appear identically to the two eyes.

The size of the image formed on the retina is equal to the size of the object multiplied by the focal length of the eye and divided by the distance of the object from the first focal point of the eye (§ 70) ; thus,

$$\text{Size of retinal image of eye } S_1 = yfP$$
$$\text{,,} \quad \text{,,} \quad \text{,,} \quad S_2 = y''fL = my'fL.$$

The small difference between the focal lengths of the eyes due to the difference between their accommodation conditions is here neglected.

If the two retinal images are to be equal in size, then the magnification produced is

$$\beta = \frac{y'}{y} = \frac{P}{mL} \quad \dots \quad \dots \quad (10)$$

Since the objective is surrounded by air $n=n'$, and its first and second focal lengths f and f' are equal.

In Fig. 175

$$\frac{1}{p} = P \qquad \frac{1}{p'} = P' \qquad \frac{1}{f} = \frac{1}{f'} = D.$$

Applying equation (3) § 39, we obtain

$$\frac{P}{B} + P'B = D \quad \dots \quad \dots \quad (11)$$

where $B = \frac{\rho'}{\rho}$.

Equation (4) § 39, and equation (10) above, give

$$\beta = \frac{P}{P'} \cdot \frac{1}{B} = \frac{P}{mL} \quad \dots \quad \dots \quad (12)$$

Hence

$$D = mL + \frac{P}{B} \quad \dots \quad \dots \quad (12a)$$

For symmetrical objectives in which the aperture diaphragm is placed at the mid-point, $B=1$, so that

$$D = mL + P \quad \dots \quad \dots \quad (13)$$

In the following examples assume $B=1$.

Ex. 105.

It is desired to photograph the landscape so that when the photograph is observed without subsequent magnification by the eye S_2 at a distance of 25 cm., it appears faithfully to reproduce the landscape. How must the objective be constructed ?

$$P = 0 \; ; \; m = 1 \; ; \; L = 4 \; Dp.$$

Hence from equation (13), $D = 4$ *Dp.* or $f = 25$ cm. If the aperture be chosen = 5 mm., then the aperture ratio is 1/50.

Ex. 106.

A cinematographic photograph which is to be enlarged 200 times and viewed from a distance of 10 m., is required to appear true to nature. The aperture ratio of the objective is to be $f/7{\cdot}5$. What will be the focal length and aperture of the objective ?

Assuming that the objects are very distant, $P = 0$; $m = 200$; $L = \frac{1}{10}$ *Dp.*; hence $D = 20$ *Dp.*, corresponding to a focal length of 5 cm., and the aperture $= \frac{50}{7 \cdot 5} = 6 \cdot 6$ mm.

Ex. 107.

A photograph of a small natural object (a flower) is to be taken, which, when viewed from a distance of 25 cms., will appear exactly the same as the original object viewed from an equal distance. What kind of objective must be used ?

$$P = 4 \; Dp. \; ; \; L = 4 \; Dp. \; ; \text{ hence } D = 4m + 4.$$

If no further enlargement is to take place, then $m = 1$ and $D = 8$ *Dp.*, corresponding to $f = \frac{100}{8} = 12{\cdot}5$ cm.

The pupil of the eye contracts (convergence of the visual axes) when viewing near objects, hence the objective aperture must be assumed small—say 3 mm.

The aperture ratio then, will be about 1/40.

If the only available objective had been one of 20 cm. focal length, the value of m would have been (from equation (13))

$$m = \frac{D - P}{L} = \frac{5 - 4}{4} = \frac{1}{4},$$

i.e. the photograph would need to be subsequently reduced 4 times.

Illumination and Depth of True-to-Nature Photographs.

Since $p = \frac{1}{P}$,

$$\beta = \frac{P}{mL} \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

and

$$D = \frac{P}{B} + mL$$

equation (7) gives

$$H = \pi c . m^2 L^2 . \rho^2 \quad \ldots \quad \ldots \quad (15)$$

Since ρ is to be equal to the radius of the pupil of the eye, this expression for illumination contains no terms depending on the objective.

From equations (14), (3a) and (6) we obtain

$$T_n = T_f = \frac{w_o}{2\rho} \quad \dots \quad \dots \quad (16)$$

142. Modern Types of Objectives.

A brief summary of the more useful types of photographic objectives is given below.*

A. Objectives which do not give Anastigmatic Flattening of the Field.

Simple or achromatic converging meniscus lenses with their concave surfaces facing the object and provided with front stops are used as landscape lenses. About 1868, Steinheil introduced his **Periscope** and **Aplanat**, which are symmetrical double objectives consisting of similar positive menisci. Later, the same firm produced the **Antiplanet**, in which the second member of the Aplanat is replaced by a compound lens consisting of a flint lens and a lens of crown glass separated from one another. All these types, however, were superseded to a great extent by objectives producing anastigmatic flattening of the field.

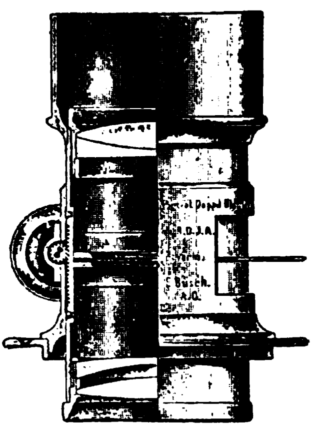

FIG. 176.

The **Petzval** objective, in spite of the fact that it is not astigmatically corrected, is still used extensively, and maintains its position as a testimonial to the ability of its designer; it has great light-transmitting power and its central definition has scarcely been surpassed. It is particularly suitable for enlarging purposes; the following

* *See* M. v. Rohr. "Theorie und Geschichte des photographischen Objektives," 1899.

See also the smaller work by Harting "Optischen Hilfsbuch für Photographierende." Berlin, 1909, p. 111 *et seq.*

table gives the constructional data of a Petzval objective as made by Busch of Rathenau (*see* Fig. 176).

Petzval Portrait Objective by Busch.*

Relative aperture $f/3$; focal length = 150 mm.

Radii.	Thicknesses and Separations.	System.	Glass.	
			n_D.	$n_F - n_C$.
$r_1 = + \ 87 \cdot 09$				
	$d_1 = 10 \cdot 0$	I	1·5181	0·00858
$r_2 = - \ 73 \cdot 23$				
	$d_2 = 2 \cdot 0$	II	1·5783	0·01409
$r_3 = + 697 \cdot 28$				
	$e_1 = 54 \cdot 6$	Air		
$r_4 = + 215 \cdot 78$				
	$d_3 = 2 \cdot 5$	III	1·5783	0·01409
$r_5 = + \ 64 \cdot 08$				
	$e_2 = 4 \cdot 0$	Air		
$r_6 = + \ 77 \cdot 16$				
	$d_4 = 8 \cdot 0$	IV	1·5152	0·00906
$r_7 = - 224 \cdot 48$				

B. Anastigmats.

In a large number of modern objectives, in addition to the fulfilment of the sine condition and the elimination of spherical aberration at least for one zone, flattening of the image is accomplished to a great extent; *i.e.*, the meridional and sagittal image points of incident oblique bundles are made to coincide approximately in the focal plane. The **Hypergon** wide-angle lens, designed by Hoegh and made by Goerz, consists of two symmetrically disposed and very strongly curved menisci; it is uncorrected for spherical aberration but is anastigmatically flattened over an angle of about 140°.

Below we give a few historical notes and examples of recent constructions (or at least of recent calculations) of anastigmatic objectives that have been produced from time to

* In the following tables of constructional data, *d* will be used to represent the lens thickness and *e* the air separation between the lenses; *b* the distance from the centre of the stop to the vertex of the nearest lens. The various components of the compound systems will be denoted by I, II, III, &c.—Trans.

time. In all cases the objectives are chromatically corrected, the sine condition is fulfilled, and spherical aberration at least for one zone is eliminated.

143. Zeiss Objectives.

Since 1886 types of glass of relatively high refractive index and relatively high dispersion have been made by Schott and Co. at Jena. The first, the Baryta flint and the Baryta crown, were immediately used with advantage in the Steinheil Aplanat type of objective. It remained, however, for Dr. P. Rudolph of Messrs. Zeiss to introduce in 1889, the second kind of glass in the making of photographic objectives. He was the first to obtain in an objective great illumination combined with anastigmatic flattening of the field. Of the numerous recent types of objectives due to Zeiss may be mentioned in particular six anastigmats which have been placed on the market by that firm.

The attention of photographers was attracted by the **Protars*** as being the first objectives of great illumination and with anastigmatic flattening of the field. This type of objective was established in 1889, and Messrs. Zeiss, who up to this time had confined their attention to microscope construction, placed the Protar on the market and commenced in 1890 the manufacture of photographic objectives. The rapid Protar lenses with apertures $f/4{\cdot}5$, $f/6{\cdot}3$, $f/7$ and $f/8$, and consisting of five lenses—a double front lens and triple back lens—have since been superseded by the **Tessar.** At present only the Protars with relative apertures $f/9$ and $f/18$ are manufactured.† They each consist of four lenses cemented together in pairs; the front lens consists of a divergent and a convergent meniscus lens, the back component of a bi-concave and a bi-convex. In the front lens, the divergent lens has the higher refractive index; in the back lens the convex component has the higher index. In the former the cemented surface is concave to the stop and is diverging; in the latter, the cemented surface is convex to the stop. This type is represented in FIG. 177. The constructional data are as follow: ‡

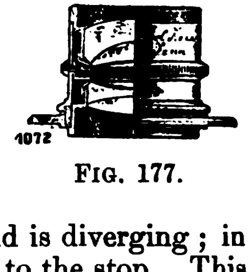

FIG. 177.

* German Patent No. 56109. April 3, 1890–1905.

† These are protected by the German Patent No. 193439; Nov. 6, 1906.

‡ From the German Patent No. 193439.

Protar, $f/12\cdot5$. $f = 100$ mm.

Radii.	Thicknesses and Separations.	System.	Glass.	
			n_D.	$n_{G'}$.
$r_1 = 17\cdot5$				
	$d_1 = 2\cdot9$	I	1·6489	1·6744
$r_2 = 5\cdot8$				
	$d_2 = 1\cdot3$	II	1·6031	1·6239
$r_3 = 18\cdot6$				
	$b_1 = b_2 = 1\cdot5$			
$r_4 = -12\cdot8$				
	$d_3 = 1\cdot1$	III	1·5154	1·5275
$r_5 = 18\cdot6$				
	$d_4 = 1\cdot8$	IV	1·6112	1·6260
$r_6 = -14\cdot3$				

The triple Protar lens* was worked out in November, 1891, and produced in the following December. In 1895 it was superseded by the rapid **quadruple Protar** lens $f/12\cdot5$ but was again introduced in an improved form† with a relative aperture of $f/12\cdot5$ in 1907. Since, however, a new glass of the requisite unusually low refractive index necessary for the complete realization of the improvement could not, on account of unforeseen difficulties, be produced regularly, permanent manufacture of the lenses could not be established. However, the improvement may be introduced in a highly corrected double objective $f/6\cdot8$, the members of which, with an aperture of $f/14$, produce good landscape or portrait lenses.

FIG. 178.

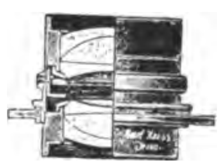

FIG. 179.

The **triple Protar** lens, as introduced in a double objective, is represented in Fig. 179. The constructional‡ data of the *back* lens are as follow :—

* English Patent No. 4692, 1891.
† German Patent No. 196734a.
‡ *See* German Patent No. 196734*a*.

Triple Protar* (Back Lens), *f/11*. $f = 100$ mm.

Radii.	Thicknesses and Separations.	System.	Glass.	
			n_D.	$n_{G'}$.
	$b = 1 \cdot 4$			
$r_1 = -15 \cdot 0$				
	$d_1 = 0 \cdot 8$	I	1·4967	1·5063
$r_2 = 36 \cdot 4$				
	$d_2 = 2 \cdot 0$	II	1·6128	1·6286
$r_3 = -6 \cdot 9$				
	$d_3 = 2 \cdot 2$	III	1·6570	1·6810
$r_4 = -15 \cdot 6$				

About the same time a Series XI*a*. was first announced, consisting of a symmetrical objective with the requisite orthoscopy for photogrammetric purposes.† The objective has an aperture of $f/8$ and is known, on account of its particular properties, as the **Orthoprotar.**

The **quadruple Protar**‡ lens. ($f/12{\cdot}5$.)—This type was introduced in 1895. It has been made uninterruptedly since that date and its position has recently been strengthened by an improvement due to Dr. Rudolph, which greatly helps the manufacture and improves the performance of the objective. The constructional data§ are as follows: (*see* Fig. 180).

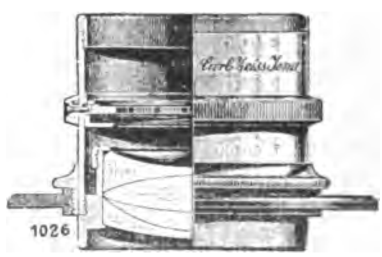

FIG. 180.

* *See* German Patent No. 196734*a*.

† That is, for the purpose of taking photographs that are to be used subsequently for measuring purposes of some kind as, for example, when surveying with the aid of photography. Such objectives must be orthoscopic.—Trans.

‡ *See* English Patent No. 19509. 1894.

§ This improvement is protected by German Patent No. 228677.

Quadruple Protar, $f/12 \cdot 5$. $f = 100$ mm.

Radii.	Thicknesses and Separations.	System.	Glass. n_D.	Glass. $n_{G'}$.
	$b = 2 \cdot 34$			
$r_1 = -12$				
	$d_1 = 0 \cdot 48$	I	1·4979	1·5074
$r_2 = 44 \cdot 2$				
	$d_2 = 1 \cdot 7$	II	1·6227	1·6308
$r_3 = -9 \cdot 3$				
	$d_3 = 1 \cdot 5$	III	1·5813	1·5952
$r_4 = -6 \cdot 6$				
	$d_4 = 0 \cdot 66$	IV	1·6275	1·6487
$r_5 = -13 \cdot 4$				

The Planar* was put on the market in 1897. In connection with this objective it was pointed out that with a comparatively simple construction, anastigmatic flattening of the image and elimination of spherical aberration was possible to an extent sufficient for weak microscope magnifications. The Planars possess a large aperture ratio and may be corrected apochromatically as desired for *reproduction objectives.*

The Planar is made symmetrical and consists of four separated lenses, of which the inner two are divergent and built up of two parts cemented together.

The Planar type is shown in Fig. 178. For particulars of the constructional data *see* the above-mentioned German patent and the work of v. Rohr "Theorie und Geschichte des photographischen Objektivs," p. 391.

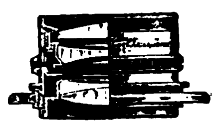

FIG. 181.

The **Tessar**† was introduced by Dr. Rudolph in 1902. It consists of three separated lenses, of which the last lens is composed of two cemented elements and is separated from the two-lens front combination by the diaphragm. The cemented surface acts convergently and is convex to the stop. The adjacent pair of surfaces of the front combination acts divergently.

* German Patent No. 92313, 14th November, 1896.
† German Patent No. 142294, April 25, 1902.

This type combines the qualities of very good central definition with anastigmatic flattening of the field which, with the high illumination of the objective, satisfies the requirements of reproduction work.

The Tessar type is shown in Fig. 181. The constructional data given in the above-mentioned patent are as follows :—

Tessar, f/5·3. Focal length = 100 mm.

Radii.	Thicknesses and Separations.	System.	Glass. n_D.	n_F.	$n_{G'}$.
$r_1 = 21\cdot5$					
	$d_1 = 3\cdot3$	I	1·61132	1·61870	1·62462
$r_2 = \pm\infty$					
	$e = 1\cdot9$	Air			
$r_3 = -74\cdot2$					
	$d_2 = 1\cdot1$	II	1·60457	1·61436	1·62252
$r_4 = 20\cdot8$					
	$b_1 = b_2 = 3\cdot0$				
$r_5 = -111\cdot3$					
	$d_3 = 1\cdot1$	III	1·52110	1·52820	1·53397
$r_6 = 25\cdot2$					
	$d_4 = 3\cdot0$	IV	1·61132	1·61895	1·62514
$r_7 = -36\cdot7$					

More recent data for the Tessars (Rudolph and Wandersleb) are published in "Zeitschrift für Instrumentenkunde," 1907, pp. 78, 79. Two forms, which approximate very closely to those already made, are given below.

Tessar, f/3·5. $f_D = 100$ mm.

Radii.	Thicknesses and Separations.	System.	Glass. n_D.
$r_1 = 29\cdot6$			
	$d_1 = 5\cdot3$	I	1·61420
$r_2 = \infty$			
	$e_1 = 7\cdot7$	Air	
$r_3 = -59\cdot1$			
	$d_2 = 1\cdot8$	II	1·61351
$r_4 = 26\cdot2$			
	$b_1 = 5\cdot3$ $b_2 = 3\cdot1$		
$r_5 = \infty$			
	$d_3 = 2\ 1$	III	1·52753
$r_6 = 27\cdot1$			
	$d_4 = 5\cdot9$	IV	1·61758
$r_7 = -41\cdot2$			

Tessar, *f*/4.5. $f_D = 100$ mm.

Radii.	Thicknesses and Separations.	System.	Glass. n_D.
$r_1 = 26 \cdot 3$			
	$d_1 = 3 \cdot 7$	I	1·61342
$r_2 = \infty$			
	$e_1 = 4 \cdot 0$	Air	
$r_3 = -58 \cdot 1$			
	$d_2 = 1 \cdot 7$	II	1·57391
$r_4 = 23 \cdot 9$			
	$b_1 = 3 \cdot 7$ $b_2 = 1 \cdot 6$		
$r_5 = -146 \cdot 7$			
	$d_3 = 1 \cdot 7$	III	1·53000
$r_6 = 22 \cdot 3$			
	$d_4 = 4 \cdot 6$	IV	1·61451
$r_7 = -36 \cdot 3$			

The **Ortho-Protar** of Rudolph for photogrammetric purposes.

In the photographic objective catalogue (1910) of Zeiss, the Ortho-Protar ($f/8$) Series IX*a*. is recommended for purposes of photogrammetry; it is a symmetrical objective which, like all symmetrical objectives, not only forms images that are entirely free from distortion when the magnification is unity, but also forms images that have extraordinarily little residual distortion in the case of infinitely distant objects viewed over large angles.

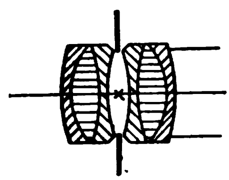

FIG. 182.

The following are the constructional data*:

Ortho-Protar, *f*/8.

Radii.	Thicknesses and Separations.	System.	Glass. n_D.
$+r_1 = -r_8 = 20 \cdot 83$			
	$d_1 = d_6 = 1 \cdot 10$	I = VI	1·62210
$+r_2 = -r_7 = 9 \cdot 15$			
	$d_2 = d_5 = 3 \cdot 64$	II = V	1·58950
$-r_3 = +r_6 = 30 \cdot 31$			
	$d_3 = d_4 = 1 \cdot 06$	III = IV	1·49833
$+r_4 = -r_5 = 19 \cdot 04$			
	$b_1 = b_2 = 1 \cdot 41$		

* German Patent No. 196734a.

The distortion* when $\beta=0$ is given by the expression (§ 48).

$$V_{\beta=0} = \frac{p_o' + \delta'}{f} \cdot \frac{\tan \omega'}{\tan \omega} - 1$$

which gives the following values on substitution :—

when $\omega = 17^{\circ}{\cdot}4$	$V_{\beta=0} = -\ 0{\cdot}12.\,^{\circ}/_{\infty}$
$\omega = 24^{\circ}{\cdot}2$	,, $= -\ 0{\cdot}28.\,^{\circ}/_{\infty}$
$\omega = 29^{\circ}{\cdot}5$	,, $= -\ 0{\cdot}25.\,^{\circ}/_{\infty}$
$\omega = 36^{\circ}{\cdot}7$	,, $= -\ 0{\cdot}0.\,^{\circ}/_{\infty}$

where the angles ω are the inclinations of the principal rays on the object side. If the distortion curve be drawn to the usual scale, the deviations from the horizontal axis would scarcely be seen ; hence in Fig. 183 the ordinates $V_{\beta=0}$ are drawn ten times their proper size.

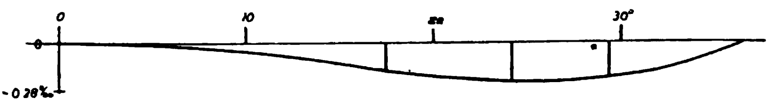

FIG. 183.

An Ortho-Protar for photogrammetry was constructed in 1902 by Zeiss ; likewise a symmetrical objective, each of whose component systems consisted of five cemented lenses.† This was not proceeded with, as the much simpler Tessars ($f/6{\cdot}3$), already being regularly manufactured, satisfied the requirements.

The curve (Fig. 183) shows that the distortion for $\beta = 0$ with the new Ortho-Protar ($f/8$) has still smaller values than in the case of the Tessar ($f/6{\cdot}3$) and is corrected for larger angles than the earlier Ortho-Protar.‡ There are two other objectives, the **Pantoscope** of Busch, and the **Hypergon** double anastigmat of Goerz,§ which, like the new Ortho-Protar, are free from distortion over their whole angular range, and are good for anastigmatic flattening ; the Ortho-Protar, however, is, in addition, spherically corrected and satisfies the sine condition for the relative aperture $f/8$, and consequently may be classed as a rapid objective which may be an important advantage for photogrammetric work.

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* *See* E. Wandersleb "Ueber die Verzeichnungsfehler photographischer Objective." Zeitschr. f. Instrumentenkunde, 1907, p. 77.

† *See* Zeitschr. f. Instrumentenkunde, 1907, p. 80.

‡ *See* Zeitschr. f. Instrumentenkunde, March 1907. Curves Nos. 18 and 39.

§ *See* Zeitschr. f. Instrumentenkunde. Curves Nos. 48 and 52.

144. Photographic Objectives of Goerz.

Goerz commenced the manufacture of photographic objectives in 1888, introducing first their **Lynkeioscope** in various series. These were calculated by Carl Moser and resembled in their form the Aplanats of Steinheil. These objectives are still manufactured, since, in spite of their lack of anastigmatic flattening of the image, they are very well adapted for many purposes on account of their brilliance.

After the death of Moser, the first **Double Anastigmat** was calculated by Emil v. Hoegh.* This has been called latterly (from 1904 onwards) the **Dagor**, in order to distinguish it from other objectives which are also called Double Anastigmats. This objective consists of two symmetrical components, each of three cemented lenses, of which the positive meniscus has a lower and the bi-convex lens a higher refractive index than the intermediate bi-concave lens. Since 1901, this objective has been made with a relative aperture of $f/6{\cdot}8$ (previously $f/7{\cdot}7$). The "**Dagor**" is in very general use as a Universal objective of a first quality type.

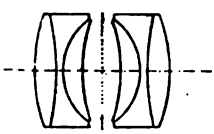


FIG. 184.

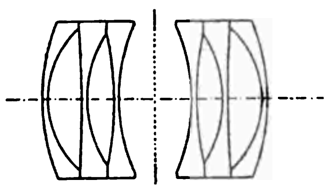


FIG. 185.

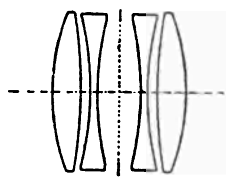


FIG. 186.

In addition to this series of objectives there has been produced for reproduction purposes another series based on the same principle and having a relative aperture of $f/11$.

Since in the double anastigmats, the half components could not be satisfactorily used individually with full aperture, v. Hoegh calculated a convertible objective† which consisted (Fig. 185) of five lenses and was well corrected. On account of its high cost and relatively small field of view (70° – 75°), the manufacture of this objective was abandoned soon afterwards.

A new objective of great illuminating power, the "**Celor**" ($f/5$) (Fig. 186) was put on the market in 1901.‡ It consisted of two symmetrical halves, each of two separated lenses, and was the first brilliant objective that could be

* German Patent No. 74437; December 20, 1892.
† English Patent 13904, 1897.
‡ German Patent No. 109283, May 27, 1898.

used for plates whose long sides were equal to the focal length. In spite of its great rapidity, the Celor *f*/5 has very small zonal error and is well corrected for chromatism.

The "**Syntor**" is of similar construction with a relative aperture of *f*/6·8.

An objective of the other extreme is the Goerz Double Anastigmat "**Hypergon**"* with a rapidity of *f*/22 (Fig. 187.) The Hypergon, consisting of two menisci, has a field of view of about 135°, over the whole extent of which it is anastigmatically flattened. Moreover, it is free from distortion and is consequently suitable for photogrammetry. On the other hand it lacks spherical correction.

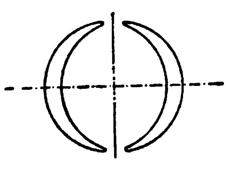


FIG. 187.

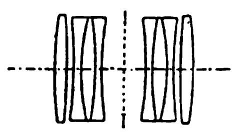


FIG. 188.

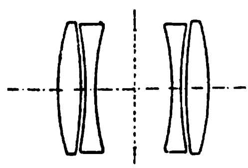


FIG. 189.

In 1902 v. Hoegh left the firm and Zschokke became head of the mathematical department. The latter, in conjunction with Franz Urban, set himself the problem of making a reproduction objective which would be suitable for three colour photography. The result of their researches was the "**Alethar**" which was placed on the market in 1903. It exhibited very good colour correction together with small spherical aberration and good flattening of the image. Unfortunately, these objectives could not be produced uniformly, owing to the fact that the individual glass plates exhibited too large deviations in their optical density.† At the expense of theoretical corrections, which were seldom realized in practice, a simple objective, the "**Artar**" which was intended to replace the "**Alethar,**" was designed later. It consists of only four separated lenses (Fig. 189), so that lenses with unequal optical densities can easily be detected and replaced by others.

The efforts to establish a convertible objective led to the "**Pantar.**"‡ The individual lenses (Pantar lenses) had an aperture of *f*/12·5. The double objective is only a little inferior to the "Dagor"; it has a somewhat smaller angle and therefore greater rapidity. (Fig. 190.)

* German Patent No. 126500, June 21, 1900.

† *See* Zeitschr. f. Instrumentenkunde, 1909; W. Zschokke, "Homogenität des optischen Glases."

‡ German Patent, No. 171369, May 11, 1904.

The patent for a very rapid objective, the **Celor** $f/3\cdot5$, was announced in 1907.* It is intended for cinematographic, sport and portrait photography as well as autochromatic

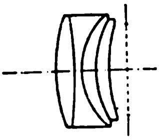


FIG. 190.

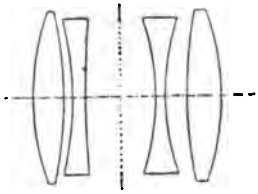


FIG. 191.

work and consists of four separated lenses, which are not arranged symmetrically about the stop (Fig. 191). The flattening of the image extends over an angle of 40° within which distortion is practically eliminated.

Dagor, $f/6.8$. $f = 100$ mm. (Fig. 184.)

Radii.	Thicknesses.	System.	Glass. n_D.	Glass. $n_{G'}$.
$r_1 = +20\cdot8$				
	$d_1 = 3\cdot0$	I	1·6122	1·6262
$r_2 = -34\cdot4$				
	$d_2 = 0\cdot9$	II	1·5481	1·5613
$r_3 = +8\cdot5$				
	$d_3 = 2\cdot5$	III	1·5120	1·5228
$r_4 = +21\cdot1$				

$\frac{\Delta}{2} = 2\cdot7$ Symmetrical.

Satz-anastigmat, $f/5\cdot5$. $f = 100$ mm. (Fig. 185.)

Radii.	Thicknesses.	System.	Glass. n_D.	Glass. $n_{G'}$.
$r_1 = +27\cdot5$				
	$d_1 = 0\cdot7$	I	1·6038	1·6174
$r_2 = +12\cdot7$				
	$d_2 = 4\cdot5$	II	1·5384	1·5497
$r_3 = -146\cdot5$				
	$d_3 = 0\cdot7$	III	1·5141	1·5256
$r_4 = +16\cdot0$				
	$d_4 = 3\cdot3$	IV	1·6103	1·6235
$r_5 = -54\cdot9$				
	$d_5 = 0\cdot7$	V	1·5137	1·5256
$r_6 = +24\cdot2$				

$\frac{\Delta}{2} = 4\cdot8$ Symmetrical.

* German Patent, No. 202083, 1907.

Celor, f/5. $f = 100$ mm. (Fig. 186.)

Radii.	Thicknesses.	System.	Glass.	
			n_D.	$n_{G'}$.
$r_1 = +29\cdot8$				
	$d_1 = 3\cdot7$	I	1·6097	1·6228
$r_2 = -64\cdot2$				
	$e_1 = 1\cdot1$	Air	1·0	1·0
$r_3 = -41\cdot2$				
	$d_2 = 1\cdot0$	II	1·5407	1·5553
$r_4 = +41\cdot2$				

$\frac{\Delta}{2} = 2\cdot9$ Symmetrical.

Hypergon, f/22. $f = 100$ mm. (Fig. 187.)

Radii.	Thicknesses.	System.	Glass.	
			n_D.	$n_{G'}$.
$r_1 = +8\cdot57$				
	$d = 2\cdot2$	I	1·5105	1·5205
$r_2 = +8\cdot63$				

$\frac{\Delta}{2} = 6\cdot9$ Symmetrical.

Alethar, f/11. $f = 100$ mm. (Fig. 188.)

Radii.	Thicknesses.	System.	Glass.	
			n_D.	$n_{G'}$.
$r_1 = +32\cdot4$				
	$d_1 = 1\cdot0$	I	1·6155	1·6297
$r_2 = -75\cdot7$				
	$e_1 = 0\cdot6$	Air	1·0	1·0
$r_3 = -37\cdot9$				
	$d_2 = 0\cdot6$	II	1·5221	1·5352
$r_4 = +19\cdot3$				
	$d_3 = 1\cdot3$	III	1·6155	1·6297
$r_5 = -31\cdot2$				
	$d_4 = 0\cdot6$	IV	1·5221	1·5352
$r_6 = +31\cdot2$				

$\frac{\Delta}{2} = 1\cdot9$

Artar, $f/9$. $f = 100$ mm. (Fig. 189.)

Radii.	Thicknesses.	System.	Glass. n_D.	Glass. $n_{G'}$.
$r_1 = +21 \cdot 3$				
	$d_1 = 2 \cdot 0$	I	1·6090	1·6222
$r_2 = -36 \cdot 9$				
	$e_1 = 0 \cdot 4$	Air	1·0	1·0
$r_3 = -29 \cdot 6$				
	$d_2 = 0 \cdot 9$	II	1·5254	1·5386
$r_4 = +22 \cdot 0$				

$$\frac{\Delta}{2} = 3 \cdot 2.$$

Pantar lens, $f/12.5$. $f = 100$ mm. (Fig. 190.)

Radii.	Thicknesses.	System.	Glass. n_D.	Glass. $n_{G'}$.
$r_1 = -14 \cdot 2$				
	$d_1 = 1 \cdot 1$	I	1·6253	1·6465
$r_2 = -8 \cdot 9$				
	$d_2 = 1 \cdot 3$	II	1·4643	1·4732
$r_3 = -6 \cdot 6$				
	$d_3 = 0 \cdot 6$	III	1·5414	1·5563
$r_4 = +35 \cdot 1$				
	$d_4 = 1 \cdot 6$	IV	1·6210	1·6349
$r_5 = -15 \cdot 1$				

Celor, $f/3.5$. $f = 100$ mm. (Fig. 191.)

Radii.	Thicknesses.	System.	Glass. n_D.	Glass. $n_{G'}$.
$r_1 = +43 \cdot 5$				
	$d_1 = 5 \cdot 2$	I	1·6141	1·6280
$r_2 = -70 \cdot 4$				
	$e_1 = 1 \cdot 6$	Air	1·0	1·0
$r_3 = -55 \cdot 9$				
	$d_2 = 2 \cdot 1$	II	1·6051	1·6260
$r_4 = +163 \cdot 9$				
	$b_1 + b_2 = 12 \cdot 0$	Air	1·0	1·0
$r_5 = -41 \cdot 7$				
	$d_3 = 1 \cdot 8$	III	1·5513	1·5672
$r_6 = +41 \cdot 7$				
	$e_2 = 3 \cdot 6$	Air	1·0	1·0
$r_7 = +78 \cdot 1$				
	$d_4 = 5 \cdot 7$	IV	1·6141	1·6280
$r_8 = -34 \cdot 4$				

145. Objectives of Voigtlaender.

Of the older objectives not corrected astigmatically, Voigtlaender still make regularly the so-called "**Euryscope**" with aperture ratio $f/4{\cdot}5$ and $f/7$, which are similar in type to the Steinheil aplanats; also two series of portrait objectives, according to Petzval and Linken-Sommer. The latter was made by Harting to satisfy the requirements of that time and was, for an aperture of $f/2{\cdot}3$, the most rapid existing objective. The construction is shown in Fig. 192.

FIG. 192.

FIG. 193.

The "**Collinear,**" calculated by Scheffer, and produced in four series, exhibited anastigmatic flattening of the image.* The **Triple Anastigmat** was calculated by Dennis Taylor and is remarkable for its performance when compared with its simplicity of manufacture (three simple separated lenses).†

The newer constructions of the last ten years are all first class creations from new calculations by Harting such as the **Apochromat-Collinear**, the **Heliar,**‡ **Dynar**‡ and **Oxyn.**

The **Apochromat-Collinear** is specially adapted for reproduction photography and on account of its freedom from secondary spectra, it is particularly suited for colour photography.§

FIG. 194.

FIG. 195.

* For constructional data *see* German Patent No. 88505.

† For constructional data *see* German Patent No. 86757. (This is the lens well known as the Cooke lens.—Trans.)

‡ *See* German Patents Nos. 124934 and 143889; also *see* Harting, "Das Heliar." Photogr. Korresp. 1902.

§ *See* Harting "Ueber die Theorie des Apochromatkollineares." Photogr. Korresp. 1901.

The constructional data of the Heliar, from the German Patent No. 143889, are given below.

Heliar. $f = 18$ cm. (Fig. 193.)

Radii.	Thicknesses.	System.	Glass. n_D	Glass. ν
$r_1 = +\ 67 \cdot 9$				
	$d_1 = 2 \cdot 8$	I	1·549	45·9
$r_2 = +\ 32 \cdot 5$				
	$d_2 = 8 \cdot 5$	II	1·614	56·2
$r_3 = +\ 197 \cdot 1$				
	$e_1 = 16 \cdot 9$	Air		
$r_4 = -\ 66 \cdot 4$				
	$d_3 = 1 \cdot 1$	III	1·537	51·6
$r_5 = +\ 57 \cdot 1$				
	$b_1 + b_2 = 9 \cdot 0$	Air		
$r_6 = +\ 130 \cdot 3$				
	$d_4 = 8 \cdot 7$	IV	1·614	56·2
$r_7 = -\ 32 \cdot 5$				
	$d_5 = 1 \cdot 1$	V	1·581	41·9
$r_8 = -\ 68 \cdot 3$				

Diameter of lens = 40·5 mm.

Dynar. $f = 12$ cm. (Fig. 194.)

Radii.	Thicknesses.	System.	Glass. n_D	Glass. ν
$r_1 = +\ 31 \cdot 9$				
	$d_1 = 4 \cdot 7$	I	1·613	56·5
$r_2 = -\ 32 \cdot 1$				
	$d_2 = 1 \cdot 0$	II	1·569	51·3
$r_3 = \infty$				
	$e_1 = 2 \cdot 9$	Air		
$r_4 = -\ 66 \cdot 3$				
	$d_3 = 0 \cdot 5$	III	1·551	45·2
$r_5 = +\ 30 \cdot 1$				
	$b_1 + b_2 = 8 \cdot 1$	Air		
$r_6 = -\ 164 \cdot 4$				
	$d_4 = 0 \cdot 5$	IV	1·569	51·3
$r_7 = +\ 25 \cdot 7$				
	$d_5 = 4 \cdot 1$	V	1·613	56·5
$r_8 = -\ 42 \cdot 6$				

Diameter of lens = 22·0 mm. Free aperture 20·0 mm.

Oxyn.* $f/4.5$ $f_D = 100$ mm. (Fig. 195.)

Radii.	Thick-nesses.	System.	Glass. n_D	$n_{G'}$	Name.
$r_1 = + \ 34{\cdot}99$					
	$d_1 = 0{\cdot}67$	I	1·54890	1·56547	Light Silicate Flint.
$r_2 = + \ 20{\cdot}83$					
	$d_2 = 4{\cdot}17$	II	1·61340	1·62736	Heaviest Baryta Crown.
$r_3 = \ \infty$					
	$e_1 = 5{\cdot}83$	Air			
$r_4 = - \ 36{\cdot}67$					
	$d_3 = 0{\cdot}42$	III	1·55019	1·56597	Light Silicate Flint.
$r_5 = + \ 28{\cdot}25$					
	$b_1 + b_2 = 3{\cdot}17$	Air			
$r_6 = - \ 266{\cdot}06$					
	$d_4 = 0{\cdot}42$	IV	1·53780	1·55143	Silicate Glass.
$r_7 = + \ 19{\cdot}16$					
	$d_5 = 6{\cdot}67$	V	1·61340	1·62736	Heaviest Baryta Crown.
$r_8 = - \ 30{\cdot}19$					

Diameter of lens = 22·3 mm.
The stop is put in the second air space.

146. The Tele-Objective.

By the term Tele-Objective, is understood a combination of a positive and a negative lens at such a distance e apart, that the combination forms a real inverted image of a distant and real object.

In Fig. 196, the two component parts L_1 and L_2 are assumed to be very thin. The positive component (the tele-positive) has its vertex at S_1, focal length f_1 and power D_1, whilst the corresponding quantities of the negative component are S_2, f_2 and D_2; the negative signs of the two latter quantities have to be taken into account.

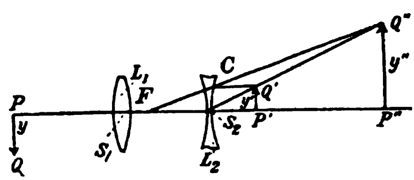


FIG. 196.

Assuming first that the second system is not present, then a real image y' of the object $PQ = y$ is formed by the system L_1. If F be the first focal point of the system L_2,

* *See* German Patent No. 154910.

an image $P''Q'' = y''$ is formed of the image $P'Q' = y'$ and its position may be determined by the usual construction applicable to thin lenses. Thus, draw $Q'C$ parallel to the axis; join FC and produce it till it cuts S_2Q' produced in $Q.''$ It is evident that the point Q'' lies to the right of L_2; hence we obtain a real image y'' when the distance S_2F is greater than S_2P', a condition which we will now consider.

Let the reciprocal of PS_1 (*i.e.* the vergence of the point P) be denoted by P;

then

$$P + \frac{1}{S_1P'} = D_1$$

and since $S_1S_2 = e$, we have

$$S_2P' = S_1P' - e = \frac{1}{D_1 - P} - e \quad \dots \qquad (1)$$

and since $S_2F = \frac{1}{D_2}$, the above condition becomes

$$\frac{1}{D_2} > \frac{1}{D_1 - P} - e \quad \dots \qquad \dots \qquad (2)$$

Moreover the quantity on the right of the expression (2) must be positive, *i.e.* the negative lens must be situated between y' and the lens L_1. Employing the simple rules of image-formation by a negative lens we have, for the position of the final image y'', the expression

$$\frac{1}{S_2P''} = \frac{1}{S_2P'} - \frac{1}{FS_2}$$

$$= \frac{D_1 - P}{1 - e\,(D_1 - P)} - D_2$$

or

$$\frac{1}{S_2P''} = \frac{D_1 - P - D_2 + eD_2\,(D_1 - P)}{1 - e\,(D_1 - P)} \dots \qquad (3)$$

If the image formation is to be "true-to-nature," *i.e.* if the object y is to be so imaged, that the image formed on the retina of an eye of refraction L when observing the photograph, is to be similar to the retinal image when the eye observes the object at a distance $\frac{1}{P}$ it can be shown* that

* *See* equation (42) p. 35 in Gleichen "Die naturgetreue photographische Abbildung"; *loc. cit.*

the following relation is true, neglecting subsequent magnification ;

$$e = \frac{L + P + D_2 - D_1}{D_2\,(D_1 - P)} \qquad \ldots \quad (3a)$$

Equation (3) then becomes

$$S_2P'' = \frac{1 - e\,(D_1 - P)}{L} \qquad \ldots \quad (4)$$

The conditions for the relative position of the two systems may now be given in a very simple form.

The condition that the right-hand side of the expression (2) must be positive leads to the statement :

$$\frac{1}{D_1 - P} - e > O$$

or

$$D_2 - eD_2\,(D_1 - P) > O.$$

Employing equation (3*a*) we obtain

$$D_1 - P - L > O$$

or finally

$$D_1 > L + P \qquad \ldots \qquad \ldots \quad (5)$$

whilst the inequality (2) gives

$$L > O$$

Thus we have this result :

In a tele-objective the power of the tele-positive must be greater than the sum of the refractions of the eye when observing the photograph and the object respectively.

Ex. 108.

The components of a tele-objective have powers $D_1 = 5\ Dp$ and $D_2 = 15\ Dp$. The object to be photographed lies at a distance of 2 m. and the photograph is to be viewed from a distance of 1 m. What must be the separation (e) of the two systems, the size of the image y'' and the distance S_2P'' of this image from the negative lens ?

From equation (3*a*) since $L = 1$ and $P = \frac{1}{2}$

$$e = \frac{23}{135}\text{ m.} = 17\text{ cm.}$$

whilst equation (4) gives $S_2P'' = 23$ cm.

From equation (10) §141, $y'' = \frac{y}{2}$.

Ex. 109.

What will be the aperture ratio of the tele-objective in Ex. 108, assuming an aperture of 1 cm. ?

Equation (X) § 33 gives the power D of the combination in terms of D_1 and D_2. Putting $-D_2$ for D_2 in this equation we have

$$D = D_1 - D_2 + eD_1D_2 = -10 + \frac{23}{135} \times 75 = 2\frac{7}{9} Dp. \qquad (6)$$

corresponding to a focal length of 36 cm.

Hence the aperture ratio is 1/36.

It must be noticed, however, that the calculated aperture ratio in the last example is not a measure of the illumination, since here it is not a case of a distant object but one at a distance of 2 m. It will be necessary to refer back to the general equation (15) § 141, viz.:

$$H = \pi c \,.\, m^2\rho^2 \,.\, L^2$$

in which, since there is no subsequent magnification, $m = 1$; hence

$$H = \pi c \,.\, \rho^2 \,.\, L^2 \quad \ldots \quad \ldots \quad (6a)$$

Were it a case of photographing a distant object, the quantities D and L would be equal, as will be seen from equation (9) § 141,

$$D = mL + \frac{P}{B}$$

when we put $m = 1$ and $P = 0$. In the latter case

$$\rho D = \rho L = \frac{M}{2}$$

where M again represents the aperture ratio.

Calling H_∞ the illumination for distant objects, these equations give

$$H_\infty = \pi c \,.\, \rho^2 \,.\, D^2 = \pi c \,.\, \frac{M^2}{4}$$

and equation (6a) may now be written

$$H = H_\infty \left(\frac{L}{D}\right)^2$$

From this last equation the difference between the illuminations of near and distant objects may be compared.

If in the last problem $L = 1$ and $D = 2\frac{7}{9} Dp.$, then

$$H = H_\infty \left(\frac{9}{25}\right)^2 = 0{\cdot}13\ H$$

The brightness of the image when focussing on an object at 2 m. is only about $\frac{1}{7}$ of the brightness when focussing on an object at infinity.

The general relations of §§ 139 and 141 may be applied when dealing with the question of depth of focus.

An advantage of the tele-objective is that *without* subsequent magnification ($m=1$) and assuming L and P given, and that two component systems of powers D_1 and D_2 are available, photographs that are true-to-nature can be taken, the corresponding separation e being derived from equation (3a) Usually, of course, the tele-objective is used for photographing distant objects, so that the power D and consequently the quantity L are made small. The photograph should then be observed from a relatively large distance; this is not usually done however, and the rather strongly magnified picture is on the contrary viewed from a position quite close up; the tele-objective being thus used, in a manner, as an enlarging apparatus.

147. Construction of modern Tele-Objectives.

The "Bis-Telar" of Busch and the "Magnar" of Zeiss.

Recently tele-objectives have been constructed in which the important variability of the focal length referred to above, has been abandoned.

It will be clear that it is impossible, with a combination consisting of a converging and a diverging component, the distance between which is variable, to correct for sharpness of image for more than one distance between the components. Consequently the idea was soon proposed to abandon the variability of focal length and obtain instead greatly improved sharpness of image—up to the full aperture of the instrument—which the old tele-objectives did not possess. The correction for spherical aberration and at the same time the fulfilment of the sine condition—whereby freedom from coma to a first approximation is guaranteed—were also effected in this type of tele-objective without particular difficulty. The removal of astigmatism and curvature of image was more difficult although these also have been practically eliminated.

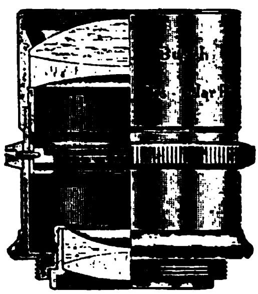


FIG. 197.

The two tele-objectives alluded to are the **Bis-Telar** made by Busch of Rathenau, worked out by Martin, and the **Magnar** of Zeiss.

With full aperture the Bis-Telar gives a sharpness which is scarcely inferior to that of a good aplanat. (Fig. 197.)

Constructional Data. Example from Patent specification.

Aperture $f/8$. $f = 240$ mm.

Radii.	Thicknesses and Separation.	System.	Glass. n_D.	Glass. $n_{G'}$.
$r_1 = + \ 29 \cdot 74$				
	$d_1 = \ 3 \cdot 84$	I	1·61358	1·62783
$r_2 = - \ 157 \cdot 38$				
	$d_2 = \ 1 \cdot 49$	II	1·61358	1·63562
$r_3 = + \ 50 \cdot 25$				
	$b_1 + b_2 = 36 \cdot 67$			
$r_4 = - \ 19 \cdot 32$				
	$d_3 = \ 1 \cdot 92$	III	1·53000	1·54100
$r_5 = + \ 480 \cdot 00$				
	$d_4 = \ 4 \cdot 8$	IV	1·51000	1·52180
$r_6 = - \ 28 \cdot 42$				

In the representation of the aberrations of oblique bundles for tele-objectives, they should not be reckoned with respect to the inclination w of the object side principal rays, but to the inclination w' of the principal rays on the image side, since a numerical relation between the angle in the image space and the corresponding angle in the object space, giving a measure of the performance of the objective, can be obtained only by repeated calculations and therefore a comparison of the different types is rendered exceptionally difficult. But here the semi-object angle w will be taken as the basis for representing the aberrations. It is desirable to adopt a larger scale of co-ordinates than is usually adopted for the ordinary objective, otherwise on account of the smallness of the angles involved in the use of the tele-objective, the curve would be too small.

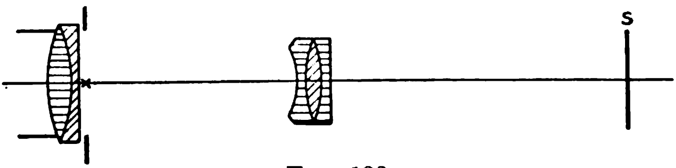


FIG. 198.

Below are given the constructional data and curves of the tele-objective "**Magnar**" $f/10$ (Fig. 198) made by Zeiss.*

* *See* German Patent No. 227112.

Magnar $f/10$.

Radii.	Thicknesses and Separation.	System.	Glass. n_D.
$r_1 = +\ 15\cdot7$			
	$d_1 = 2\cdot1$		
$r_2 = -\ 14\cdot3$			
	$d_2 = 0\cdot9$		
$r_3 = +\ 114\cdot0$			$1\cdot59133$
	$b_1 = 0\cdot3 \quad b_2 = 21\cdot1$		
$r_4 = -\ 7\cdot7$			and
	$d_3 = 0\cdot5$		
$r_5 = +\ 11\cdot1$			$1\cdot62350$
	$d_4 = 1\cdot6$		
$r_6 = -\ 11\cdot1$			
	$d_5 = 0\cdot9$		
$r_7 = -\ 77\cdot2$			

The aberrations of the bundles parallel to the axis for the D lines are represented in the usual way in Fig. 199. In the curves showing the astigmatic error (Fig. 200) and the distortion (Fig. 201), a length of 20 mm. in the ordinate corresponds to an object-side inclination w of one degree as compared with the usual scale of 4 mm. to one degree. The curves (Figs. 200 and 201) are drawn for the angular field over which the Magnar $f/10$ is employed in practice. It will be seen that within this angle the astigmatism scarcely exceeds 1·5% of the focal length, that flattening of the image is almost attained, and that distortion lies within the amount which is present in convertible lenses with front stop for the angular extent over which the latter can be usefully employed. In the present case the distortion is of the cushion shape, whilst with the convertible objectives it is barrel shaped. The aberrations for a bundle parallel to the axis (Fig. 199) are very moderate for the aperture $f/10$.

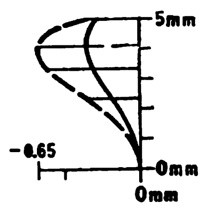


FIG. 199.

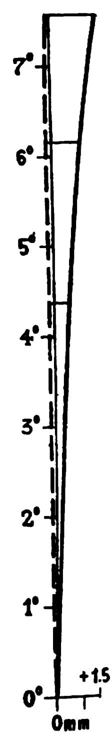


FIG. 200.

In Fig. 198, besides a section of the objective ($f = 100$ mm), the position of the plate for distant objects is indicated. It will be seen that the distance between the

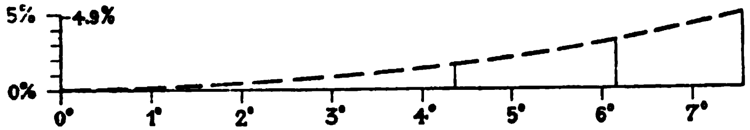


FIG. 201.

front surface of the whole system and the plate is only about one-half the focal length, whilst the distance between the vertex of the second lens and the plate is only slightly more than one-quarter of the focal length. The reduction of the axial length, which constitutes the principal advantage of the tele-objective over ordinary objectives, since it allows of the use of longer focal lengths with shorter camera extension, is thus effected to a fair extent. In a rigid examination of the performance of different tele-objectives from the diagrams representing their corrections, particular notice should be taken of the reduction of the axial length, since the further this length is diminished the more rapidly do the difficulties in the removal of the various faults increase.

References.

O. Lummer.—" Contributions to Photographic Optics," translated by S. P. Thompson. Macmillan, 1900.

Although a large number of works on the subject of photographic optics has been published in this country, the above is very valuable and contains numerous references to standard works.—Trans.

Appendix on Photographic Objectives.

In order to supply certain deficiencies in this work it is desirable to augment the text by a useful reference list, classification and bibliography of photographic objectives. In no place do lenses of British origin receive consideration, with the exception of the Cooke lens under the name of "Triple Anastigmat." Further, only modern objectives which have appeared since the publication of the author's " Lehrbuch der Geometrischen Optik " are discussed.

The following list gives the commercial name of every lens with the name of the designer. Where necessary the manufacturer's name is added in brackets. As far as possible the Patent number and date are included. A brief bibliography gives the books in

which a description of most of the objectives will be found. In the list of objectives references to such books are indicated by block letters corresponding with those in the bibliography, the numbers following referring to the page. The constructional data are placed alongside each diagram.

Photographic objectives may be divided roughly into two main groups: objectives designed prior to 1886, the year of the introduction of the Jena glasses, and those since that year. In the following list the lenses of the first group are represented by Nos. 1–35, whilst the second group includes Nos. 40 to the end. A minor group of transition lenses is also represented in Nos. 36–39. With regard to the first group, owing to the limited optical properties of the older glasses, anastigmatic flattening was not obtained, whilst the second and intermediate groups are composed almost entirely of anastigmats.

These two main groups may then be subdivided into smaller groups according to design and utility. The members of these subgroups are further arranged in chronological order. The following table shows the various subdivisions :—

1st Group.—Objectives without anastigmatic flattening (older glasses).

(*a*) Single Achromatic Lenses (Landscape Lenses), Nos. 1–11.
(*b*) Unsymmetrical Doublets (Earlier Aplanats and Antiplanets, also Petzval Portrait), Nos. 12–17.
(*c*) Symmetrical Doublets (Aplanats, Rectilinears, Wide Angle, &c.), Nos. 18–30.
(*d*) Triplet Combinations, Nos. 31–35.

Intermediate Group of objectives with anastigmatic flattening (Jena glasses employed), Nos. 36–39.

2nd Group.—Objectives with anastigmatic flattening.

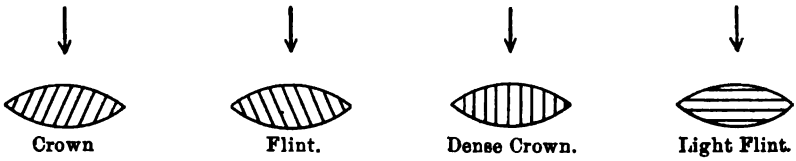
German—
(*a*) Zeiss Objectives, Nos. 40–49.
(*b*) Goerz Objectives, Nos. 50–59.
(*c*) Steinheil Objectives, Nos. 60–62.
(*d*) Voigtlaender Objectives, Nos. 63–67.

British—
(*e*) Cooke Lenses (H. D. Taylor), Nos. 68–73.
(*f*) Dallmeyer, Nos. 74–77.
(*g*) Ross, Nos. 78–81.
(*h*) Aldis, Nos. 82–84.
(*i*) Beck, Nos. 85–86.

In the diagrams the light is supposed to travel downwards in the direction of the arrow.

The standard shading of von Rohr is adopted, with an additional shading for light flint glass.

The following is the scheme for shading :—



Bibliography.

A. M. VON ROHR : Theorie und Geschichte des photographischen Objektivs. Berlin. 1899.

B. J. M. EDER : Die photographischen Objektive, ihre Eigenschaften und Prüfung. Ausführliches Handbuch der Photographie. Halle. 1883.

C. A. GLEICHEN : Lehrbuch der geometrischen Optik. Leipzig and Berlin. 1902.

D. O. LUMMER : Photographic Optics. Translated by S. P. Thompson. Macmillan & Co. London. 1900.

E. J. TRAILL TAYLOR : Optics of Photography and Photographic Lenses. Third edition with additional chapter on Anastigmatic Lenses by P. F. Everitt. Whittaker & Co. London. 1904.

F. H. SCHRÖDER : Die Elemente der photographischen Optik. Berlin. 1891.

G. H. DENNIS TAYLOR : System of Applied Optics. Macmillan. London. 1906.

H. MONKHOFEN : Optique photographique. Paris. 1873.

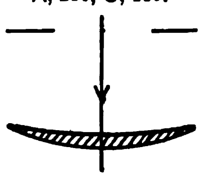
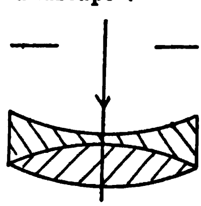
K. A. GLEICHEN : Theory of Modern Optical Instruments (the present translation).

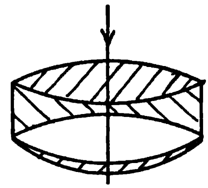
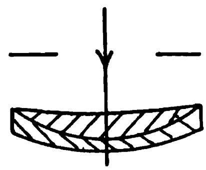
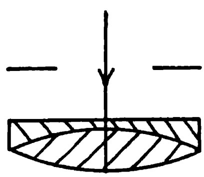
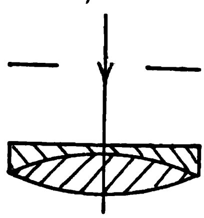
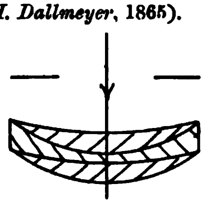
FIRST GROUP:

Objectives without Anastigmatic Flattening

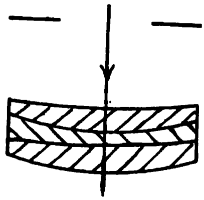
(OLDER GLASSES).

(*a*) Single Achromatic Lenses (Landscape Lenses).

Separations.	Radii.		Glass. Kind.	Constants.
$f = 560$ mm. $b = 70$ $d = 0$	$r_1 = -286{\cdot}72$ $r_2 = -143{\cdot}36$	**1. Landscape** (*Wollaston*). Phil. Trans. 2,370. A, 230, C, 469. 	Crown	$n_D = 1{\cdot}505$
General Scheme.		**2. Landscape** (*Chevalier*). 	Flint Crown	

Separations.	Radii.		Glass. Kind.	Constants.
J. Traill Taylor's Analysis.		**3. Double Periscopic** (*Goddard*, 1859). E, 46. 	Crown Flint Air Crown	
No data in Specification.		**4. Aplanatic Landscape** (*Grubb*). B.P. 2574. 1857. D, 44, E, 44. 	Crown Flint	
$f_D = 312\cdot2$ mm. $b = 40$ $d_1 = 2\cdot3$ $d_2 = 9$	$\rho = 26\cdot1$ $r_1 = \infty$ $r_2 = 99\cdot2$ $r_3 = 85\cdot6$	**5. Landscape** (*Busch*). B, 69, C, 469. 	Flint Crown	n_D 1·6027 1·5282
$f_D = 190\cdot0$ mm. $b = 20$ $d_1 = 2\cdot0$ $d_2 = 6\cdot5$	$\rho = 20$ $r_1 = -1423\cdot0$ $r_2 = 78\cdot9$ $r_3 = -78\cdot9$	**6. Landscape** (*Goerz*). C, 469. 	Flint Crown	n_D 1·610 1·530
$f_D = 6\cdot96$ ins. $d_1 = 0\cdot230$ $d_2 = 0\cdot050$ $d_3 = 0\cdot151$	$r_1 = -4\cdot29$ $r_2 = -1\cdot20$ $r_3 = -3\cdot75$ $r_4 = -1\cdot8$	**7. Wide Angle Landscape** (*J. H. Dallmeyer*, 1865). E, 44. 	Crown Flint Crown	n_D 1·5146 1·574 1·517

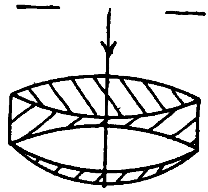
8. Rapid Landscape
(*J. H. Dallmeyer*, 1884). C, 470.



Separations.	Radii.	Glass. Kind.	Constants n_D
$f_D = 100$.	$\rho = 4 \cdot 5$		
$b = 10 \cdot 0$	$r_1 = -60 \cdot 4$	Crown	1·521
$d_1 = 3 \cdot 0$	$r_2 = -17 \cdot 3$	Flint	1·581
$d_2 = 1 \cdot 0$	$r_3 = -48 \cdot 1$	Crown	1·514
$d_3 = 1 \cdot 5$	$r_4 = -25 \cdot 6$		

9. Landscape (*Wray*, 1886).
10. Wide Angle Landscape (*Voigtlaender*, 1888).

11. Rectilinear Landscape
(*T. R. Dallmeyer*).
B.P. 1583. 1888. E. 47.



Radii	Kind	n_D
$\rho = 2 \cdot 9$ ins.		
$r_1 = +1 \cdot 558$	Flint	1·574015
$r_2 = -1 \cdot 558$	Crown	1·514591
$r_3 = +3 \cdot 342$	Air	1·0
$r_4 = -6 \cdot 001$	Crown	1·51714
$r_5 = -3 \cdot 489$		

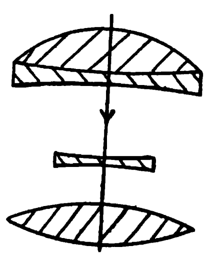
(*b*) Unsymmetrical Doublets (Earlier Aplanats and Antiplanets).

12. Portrait Aplanat
(*Steinheil*, 1874). B.P. 1124.

Separations	Radii	Kind	n_D	$n_{G'}$
$f = 9'' \ 8\frac{1}{4}'''$.	$r_1 = +45'' \cdot 8$			
$d_1 = 2'' \cdot 75$	$r_2 = +28'' \cdot 5$	F.	1·57402	1·59010
$d_2 = 4'' \cdot 84$	$r_3 = +200'' \cdot 2$	C.	1·51518	1·52530
$d_3 = 60'' \cdot 5$	$r_4 = -200'' \cdot 2$	Air	1·0	
$d_4 = 4'' \cdot 84$	$r_5 = -28'' \cdot 1$	C.	1·51518	1·52530
$d_5 = 2'' \cdot 75$	$r_6 = -61'' \cdot 7$	F.	1·57402	1·59010

13. Group Aplanat (*Steinheil*, 1879).

14. Portrait Antiplanet
(*Steinheil*). B.P. 1602. 1881.

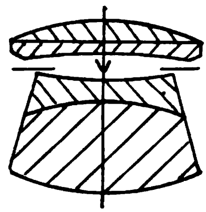


Separations	Radii	Kind	n_D	$n_{G'}$
$f = 100$.	$r_1 = +72 \cdot 24$			
$d_1 = 14 \cdot 7$	$r_2 = -240 \cdot 8$	C.	1·51705	1·53250
$d_2 = 5 \cdot 35$	$r_3 = +535 \cdot 1$	L.F.	1·57710	1·60229
$d_3 = 26 \cdot 7$	$r_4 = -144 \cdot 9$	Air	1·0	1·0
$d_4 = 5 \cdot 35$	$r_5 = +60\ 95$	L.F.	1·57710	1·60229
$d_5 = 13 \cdot 9$	$r_6 = +103 \cdot 5$	Air	1·0	1·0
$d_6 = 16 \cdot 1$	$r_7 = -96 \cdot 5$	C.	1·51705	1·53250

15. Group Antiplanet

(*Steinheil*).

G.P. 16354. 1881. C, 472, D, 61.

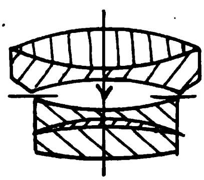


$f_D = 100.$

Separations.	Radii.	Glass. Kind.	Constants. n_D	n_G
	$r_1 = + 26·6$			
$d_1 = 2·0$		L.F.	1·57710	1·60229
	$r_2 = - 119·1$			
$d_2 = 1·7$		C.	1·51705	1·53250
	$r_3 = + 40·7$			
$d_3 = 1·7$		Air	1·0	
	$r_4 = - 40·7$			
$d_4 = 1·3$		L.F.	1·57710	1·60229
	$r_5 = + 14·3$			
$d_5 = 10·2$		C.	1·51705	1·53250
	$r_6 = - 29·9$			

16. Rapid Antiplanet

(*Steinheil*, 1893). C, 473, G, 332.



$f_D = 100.$

Separations.	Radii.	Kind.	n_D
	$r_1 = + 22·7$		
$d_1 = 2·95$		D.B.C.	1·610
	$r_2 = - 56·8$		
$d_2 = 0·91$		C.	1·518
	$r_3 = + 26·8$		
$d_3 = 4·55$		Air	1·0
	$r_4 = - 26·8$		
$d_4 = 0·91$		F.	1·578
	$r_5 = + 13·5$		
$d_5 = 1·48$		C.	1·518
	$r_6 = + 27·8$		
$d_6 = 3·2$		F.	1·578
	$r_7 = - 25·8$		

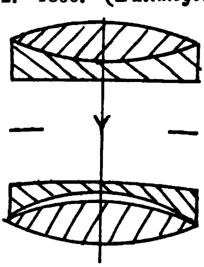
Petzval Portrait.

(*v. Rohr.*)

17. Petzval Portrait

(*Voigtlaender*, 1841).

Various modifications, *see* A, 215, 275, B, 119, C, 478, D, 59, E, K, 267. *See also* B.P. 2502. 1866. (*Dallmeyer.*)



$f_D = 100.$

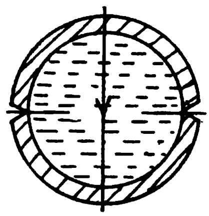
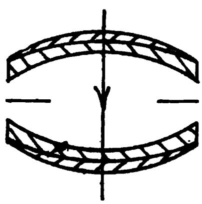
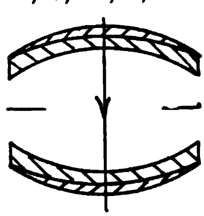
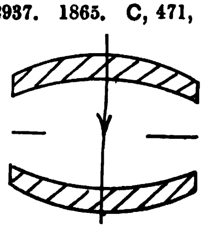
Separations.	Radii.	Kind.	n_D
	$r_1 = + 52·9$		
$d_1 = 5·8$		C.	1·517
	$r_2 = - 41·4$		
$d_2 = 1·5$		F.	1·575
	$r_3 = + 436·2$		
$d_3 = ·6$		Air	1·0
	$r_4 = + 104·8$		
$d_4 = 2·2$		F.	1·575
	$r_5 = + 36·5$		
$d_5 = 0·7$		Air	1·0
	$r_6 = + 45·5$		
$d_6 = 3·6$		C.	1·517
	$r_7 = - 149·5$		

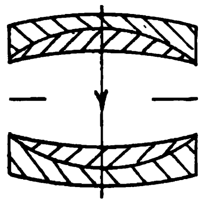
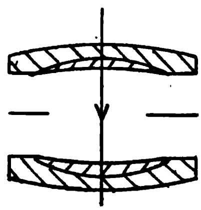
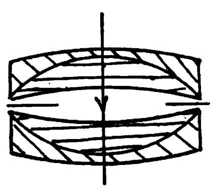
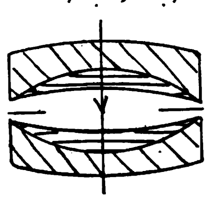
(*c*) **Symmetrical Doublets** (Aplanats, Rectilinears, Wide Angle, &c.).

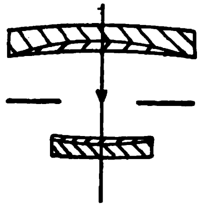
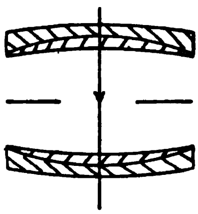
18. —— (*J. W. Draper*, 1839).

19. —— (*T. Davidson*, 1841). E, 70.

20. —— (*G. S. Cundell*, 1844). Phil. Mag., Oct., 1844.

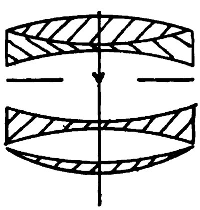
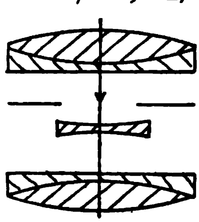
Separations.	Radii.		Glass. Kind.	Constants.
Surfaces Concentric and Symmetrical.		**21. Panorama Lens** (*T. Sutton*, 1859). B, 141, C, 478. 	Crown Water Crown	
$f_D = 100$. $d_1 = 1\cdot5$ $d_2 = 0\cdot7$ $d_3 = 22\cdot4$ Symmetrical.	$r_1 = +13\cdot4$ $r_2 = +22\cdot8$ $r_3 = +15\cdot4$	**22. Spherical (Globe) Lens** (*C. C. Harrison*). B.P. 2496. 1860. C, 478, H, 130. 	C. F. Air Symmetrical.	n_D 1·53 1·60 1·0
$f_D = 200\cdot7$. $d_1 = 3\cdot6$ $d_2 = 0\cdot4$ $d_3 = 25\cdot4$ Symmetrical.	$r_1 = +15\cdot389$ $r_2 = +21\cdot474$ $r_3 = +15\cdot759$	**23. Pantoscope** (*Busch*, 1860). B, C, 478, D, 73. 	C. F. Air Symmetrical.	n_D 1·5331 1·6079 1·0
$f_D = 100$. $d_1 = 3\cdot1$ $d_2 = 7\cdot1$ Symmetrical.	$r_1 = +20\cdot3$ $r_2 = +24\cdot3$	**24. Periscope** (*Steinheil*). B.P. 2937. 1865. C, 471, D, 73. 	Crown Air Symmetrical.	1·51072 1·0

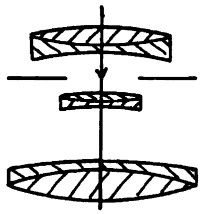
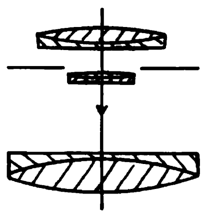
Separations. / Radii.		Glass. Kind. / Constants.
$f_D = 302\cdot5$. $r_1 = +\ 72\cdot84$ $d_1 = 2\cdot2$ $r_2 = +\ 85\cdot82$ $d_2 = 8\cdot5$ $r_3 = +\ 112\cdot0$ $d_3 = 59\cdot0$ Symmetrical.	**25. Euryscope** (*Voigtlaender*). C, 416. (Results of Trigonometrical Computation.) 	n_D / ν B.L F. 1·56772 53·4 B.S.C. 1·50902 63·9 Air 1·0 Symmetrical.
$f_D = 100$. $r_1 = +\ 24\cdot6$ $d_1 = 0\cdot8$ $r_2 = +\ 9\cdot8$ $d_2 = 1\cdot7$ $r_3 = +\ 34\cdot0$ $d_3 = 17\cdot0$ Symmetrical.	**26. Universal Aplanat** (*Steinheil*, 1860). C, 471. 	n_D F. 1·61507 C. 1·58761 Air 1·0 Symmetrical.
$f_D = 100$. $r_1 = +\ 19\cdot0$ $d_1 = 0\cdot9$ $r_2 = +\ 8\cdot2$ $d_2 = 3\cdot5$ $r_3 = +\ 23\cdot1$ $d_3 = 2\cdot4$ Symmetrical.	**27. Landscape Aplanat** *Steinheil* (*Ross in England*). 1860. C, 471. 	n_D F. 1·61588 L.F. 1·58027 Air 1·0 Symmetrical.
$f_D = 100$. $r_1 = +\ 11\cdot5$ $d_1 = 1\cdot2$ $r_2 = +\ 6\cdot5$ $d_2 = 0\cdot8$ $r_3 = +\ 17\cdot3$ $d_3 = 1\cdot7$ Symmetrical.	**28. Wide Angle Aplanat** (*Steinheil*, 1860). C, 472. 	n_D F. 1·61317 L.F. 1·57709 Air 1·0 Symmetrical.

Separations.	Radii.		Glass.	
			Kind.	Constants.
See Patent Specification.		**29. Wide Angle Rectilinear** (*J. H. Dallmeyer*). B.P. 2502. 1866. 		
No data in Specification.		**30. Rapid Rectilinear** (*J. H. Dallmeyer*, 1867). 		

(*d*) Triplet Combinations.

31. —— (*Andrew Ross*, 1841).

		32. Combination Landscape (*J. T. Goddard*, 1859). E, 63. 		
		33. Symmetrical Triplet (*T. Sutton*, 1860). E, 68. 		

Separations.	Radii.		Glass. Kind.	Glass. Constants.
		34. Triple Achromatic (*J. H. Dallmeyer* 1861). E, 63. 		
		35. Actinic Triplet (*T. Ross*, 1861). 		

Intermediate Group of Objectives with Anastigmatic Flattening.

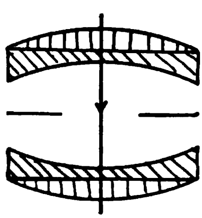
(NEWER GLASSES.)

36. Anastigmat (*Millenzei*, 1887).

37. —— (*Gray*).

38. Anastigmat Pantoscope. *Miethe* (*Hartnack*). 1888. D, 75.

39. Concentric. *H. Schröder* (*Ross*). B.P. 5194. 1888. First time on market 1892. C, 480, D, 56, 75, F, 197.



Separations.	Radii.	Glass. Kind.	Glass. Constants. n_D
$f_D = 100.$			
	$r_1 = +11 \cdot 1$		
$d_1 = 1 \cdot 4$	$r_2 = \infty$	D.B.C.	1·60
$d_2 = 0 \cdot 4$	$r_3 = +10 \cdot 2$	L.F.	1·53
$d_3 = 7 \cdot 68$	$\rho = 5 \cdot 5$		
Symmetrical.		Symmetrical.	

SECOND GROUP.

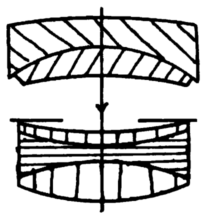
Objectives with Anastigmatic Flattening.

(*a*) By Zeiss.

40. Protar IIa (*Rudolph*).
B.P. 6028, G.P. 56109. 1890.
C, 488, D, 63, E, 184, K 268.

$f_D = 100.$

Separations.	Radii.
	$r_1 = +18·0$
$d_1 = 4·5$	
	$r_2 = +\ 8·4$
$d_2 = 3·2$	
	$r_3 = +20·1$
$d_3 = 4·92$	
	$r_4 = -23·0$
$d_4 = 1·4$	
	$r_5 = -15·2$
$d_5 = 1·0$	
	$r_6 = +21·9$
$d_4 = 2·5$	
	$r_7 = -25·6$

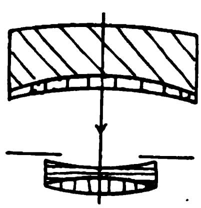


Glass. Kind.	Constants n_D
F.	1·60644
C.	1·51750
Air	1·0
D.B.C.	1·60925
F.	1·51338
C.	1·60925

41. Protar IIIa (*Rudolph*, 1891).
Improved G.P. 19349. 1906.
D, 63, E, 184, K 269.

$f_D = 100.$

Separations.	Radii.
	$r_1 = +17·5$
$d_1 = 2·9$	
	$r_2 = +\ 5·8$
$d_2 = 1·3$	
	$r_3 = +18·6$
$d_3 = 3·0$	
	$r_4 = -12·8$
$d_4 = 1·1$	
	$r_5 = +18·6$
$d_5 = 1·8$	
	$r_6 = -14·3$



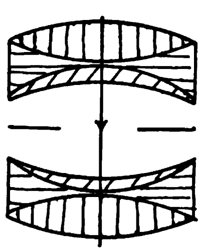
Kind	n_D	$n_{G'}$
F.	1·6489	1·6741
D.B.C.	1·6031	1·6239
Air	1·0	1·0
F.	1·5154	1·5275
D.B.C.	1·6112	1·6260

42. Triple Protar (*Rudolph*).
B.P. 4692. 1891. Improved
G.P. 196734a. 1907.
A, 368, C, 488, D, 488, K, 270.

$f_D = 100.$

Separations.	Radii.
$b = 10$	
	$r_1 = -14·2$
$d_1 = 1·7$	
	$r_2 = -\ 5·5$
$d_2 = 0·5$	
	$r_3 = +20·7$
$d_3 = 1·4$	
	$r_4 = -12·6$

Back Lens.

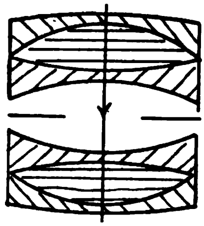


Kind	n_D	ν
Si. C.	1·52246	46·4
B.L.F.	1·56724	41·5
D.B.C.	1·6112	44·5

Back Lens.

43. Convertible Protar VIa.

(*See* 42.)



Separations.	Radii.
$f_D = 100.$	
$b = 1·4$	
	$r_1 = -15·0$
$d_1 = 0·8$	
	$r_2 = +36·4$
$d_2 = 2·0$	
	$r_3 = -\ \ 6·9$
$d_3 = 2·2$	
	$r_4 = -15·6$

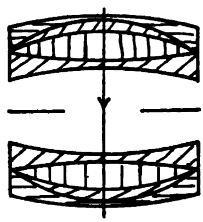
Back Lens.

Glass. Kind.	Constants. n_D	$n_{G'}$
Si.C.	1·4967	1·5063
Si.F.	1·6128	1·6286
F.	1·6570	1·6810

Back Lens.

44. Quadruple Protar VII.

Rudolph (*Ross in England*).
B.P. 19509. 1894.
Improved G.P. 228677.
C, 489, D, 64, 81, E, 188, K, 270.



Separations.	Radii.
$f_D = 100.$	
$b = 1·9$	
	$r_1 = -14·7$
$d_1 = 1·3$	
	$r_2 = +23·4$
$d_2 = 1·7$	
	$r_3 = -11·2$
$d_3 = 1·7$	
	$r_4 = -\ \ 7·4$
$d_4 = 1·3$	
	$r_5 = -15·4$

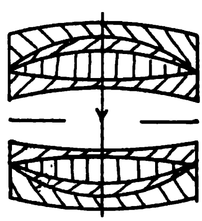
Back Lens.

Kind.	n_D
C.	1·51743
D.B.C.	1·61002
C.	1·51156
F.	1·58254

Back Lens.

45. Convertible Protar VIIa

(*Rudolph*). *See* 44.



Separations.	Radii.
$f_D = 100.$	
$b = 2·34$	
	$r_1 = -12·0$
$d_1 = 0·48$	
	$r_2 = +44·2$
$d_2 = 1·70$	
	$r_3 = -\ \ 9·3$
$d_3 = 1·50$	
	$r_4 = -\ \ 6·6$
$d_4 = 0·66$	
	$r_5 = -13·4$

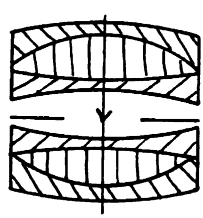
Back Lens.

Kind.	n_D	$n_{G'}$
C.	1·4979	1·5074
D.B.C.	1·6227	1·6308
C.	1·5813	1·5952
F.	1·6275	1·6487

Back Lens.

46. Ortho Protar

(*Rudolph*, 1902).
Improved G.P. 196734a. 1907.
K, 273.



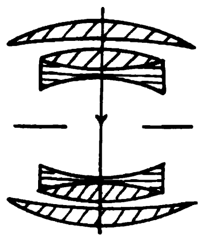
Separations.	Radii.
$f_D = 10.$	
	$r_1 = +20·83$
$d_1 = 1·10$	
	$r_2 = +\ \ 9·15$
$d_2 = 3·64$	
	$r_3 = -30·31$
$d_3 = 1·06$	
	$r_4 = +19·04$
$b = 1·41$	

Symmetrical.

Kind.	n_D
F.	1·62210
D.B.C.	1·58950
C.	1·49833
Air	1·0

Symmetrical.

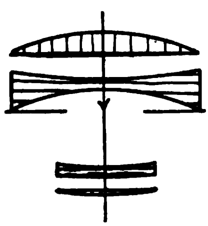
47. Planar (*Rudolph*, 1897).
A, 391, C, 489, D, 82,
E, 91, K, 271.
Also Apochromat Planar, 1903.



$f_D = 100.$

Separations.	Radii.	Glass. Kind.	Constants. n_D
	$r_1 = +\ 32{\cdot}0$		
$d_1 = 5{\cdot}0$		C.	1·56785
	$r_2 = +\ 144{\cdot}1$		
$d_2 = 0{\cdot}27$		Air	1·0
	$r_3 = +\ 37{\cdot}6$		
$d_3 = 5{\cdot}3$		C.	1·56768
	$r_4 = -\ 45{\cdot}5$		
$d_4 = 2{\cdot}7$		L.F.	1·57087
	$r_5 = +\ 20{\cdot}5$		
$b = 5{\cdot}9$			
Symmetrical.		Symmetrical.	

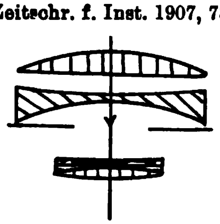
48. Unar. *Rudolph* (*Ross in England*). B.P. 24089. 1899.
C, 495, D, 83, E, 192.



$f_D = 100.$

Separations.	Radii.	Kind.	n_D	$n_{G'}$
	$r_1 = +\ 20{\cdot}7$			
$d_1 = 3{\cdot}5$		D.B.C.	1·59119	1·60344
	$r_2 = -\ 117{\cdot}4$			
$d_2 = 2{\cdot}1$		Air	1·0	1·0
	$r_3 = -\ 46{\cdot}3$			
$d_3 = 1{\cdot}4$		L.F.	1·57970	1·59779
	$r_4 = +\ 21{\cdot}1$			
$d_4 = 2b = 4{\cdot}1$		Air	1·0	1·0
	$r_5 = -\ 84{\cdot}5$			
$d_5 = 1{\cdot}4$		L.F.	1·51147	1·52159
	$r_6 = -\ 46{\cdot}3$			
$d_6 = 0{\cdot}1$		Air	1·0	1·0
	$r_7 = -\ 172{\cdot}6$			
$d_7 = 3{\cdot}5$		D.B.C.	1·61091	1·62469
	$r_8 = -\ 31{\cdot}4$			

49. Tessar (*Rudolph*).
G.P. 142294. 1902. B.P. 13061.
E, 193, K, 271,
Zeitschr. f. Inst. 1907, 78.



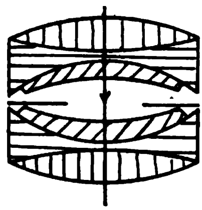
$f_D = 100.$

Separations.	Radii.	Kind.	n_D	$n_{G'}$
	$r_1 = +\ 21{\cdot}5$			
$d_1 = 3{\cdot}3$		D.B.C.	1·61132	1·62462
	$r_2 = \infty$			
$d_2 = 1{\cdot}9$		Air	1·0	1·0
	$r_3 = -\ 74{\cdot}2$			
$d_3 = 1{\cdot}1$		F.	1·60457	1·62252
	$r_4 = +\ 20{\cdot}8$			
$d_4 = 2b = 6{\cdot}0$		Air	1·00	1·0
	$r_5 = -\ 111{\cdot}3$			
$d_5 = 1{\cdot}1$		L.F.	1·52110	1.53391
	$r_6 = +\ 25{\cdot}2$			
$d_6 = 3{\cdot}0$		D.B.C.	·61132	1·62514
	$r_7 = -\ 36{\cdot}7$			

b. By Goerz.

50. Dagor. Double Anastigmat., Ser. III, IV (*Hoegh*).
G.P. 74437. 1892. Improved, 1904.
C, 494, D, 77, E, 186, K, 275.

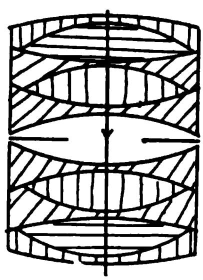
Separations.	Radii.
$f_D = 100.$	
	$r_1 = +21\cdot8097$
$d_1 = 3\cdot6397$	
	$r_2 = -33\cdot8187$
$d_2 = 1\cdot1432$	
	$r_3 = +\ 8\cdot5392$
$d_3 = 2\cdot1986$	
	$r_4 = +22\cdot2043$
$d_4 = 5\cdot4510$	
Symmetrical.	



Glass. Kind.	Constants. n_D	$n_{G'}$
D.B.C.	$1\cdot6125$	$1\cdot62635$
B.L.F.	$1\cdot5478$	$1\cdot56101$
Si.C.	$1\cdot5117$	$1\cdot52266$
Air	$1\cdot0$	$1\cdot0$
Symmetrical.		

51. Convertible Anastigmat. II (*Hoegh*).
B.P. 13094. 1897. D, 77, K, 275.

Separations.	Radii.
$f_D = 100.$	
	$r_1 = +\ 27\cdot5$
$d_1 = 0\cdot7$	
	$r_2 = +\ 12\cdot7$
$d_2 = 4\cdot5$	
	$r_3 = +146\cdot5$
$d_3 = 0\cdot7$	
	$r_4 = +\ 16\cdot0$
$d_4 = 3\cdot3$	
	$r_5 = -\ 54\cdot9$
$d_5 = 0\cdot7$	
	$r_6 = +\ 24\cdot2$
$d_6 = 9\cdot6$	
Symmetrical.	

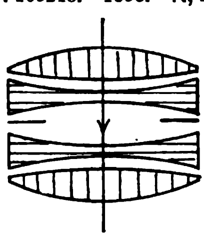


Kind	n_D	$n_{G'}$
D.B.C.	$1\cdot6038$	$1\cdot6174$
F.	$1\cdot5384$	$1\cdot5497$
C.	$1\cdot5141$	$1\cdot5256$
D.B.C.	$1\cdot6108$	$1\cdot6235$
C.	$1\cdot5137$	$1\cdot5256$
Air	$1\cdot0$	$1\cdot0$
Symmetrical.		

52. Double Anastigmat. IIa (*Hoegh*). B.P. 2894. 1892. D, 77, E, 189.

53. Celor (*Hoegh*).
G.P. 109213. 1898. K, 275.

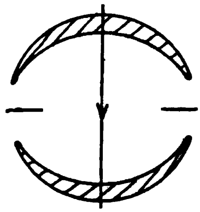
Separations.	Radii.
$f_D = 100.$	
	$r_1 = +29\cdot8$
$d_1 = 3\cdot7$	
	$r_2 = -64\cdot2$
$d_2 = 1\cdot1$	
	$r_3 = -41\cdot2$
$d_3 = 1\cdot0$	
	$r_4 = +41\cdot2$
$d_4 = 5\cdot8$	
Symmetrical.	



Kind	n_D	$n_{G'}$
D.B.C.	$1\cdot6097$	$1\cdot6228$
Air	$1\cdot0$	$1\cdot0$
F.	$1\cdot5407$	$1\cdot5553$
Air	$1\cdot0$	$1\cdot0$
Symmetrical.		

54. Hypergon (*Hoegh*).

G.P. 126500. 1900. K, 276.



$f_D = 100.$

Separations.	Radii.
	$r_1 = +\,8\cdot57$
$d_1 = 2\cdot2$	
	$r_2 = +\,8\cdot63$
$d_2 = 13\cdot8$	

Symmetrical.

Glass.

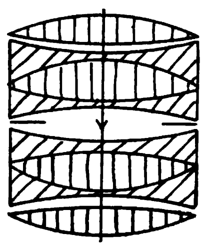
Kind.	Constants. n_D	$n_{G'}$
C.	1·5105	1·5205

Symmetrical.

55. Syntor (*Hoegh*). K, 270.

56. Alethar (*Zschokke*, 1903).

K, 276.



$f_D = 100.$

Separations.	Radii.
	$r_1 = +\,32\cdot4$
$d_1 = 1\cdot0$	
	$r_2 = -\,75\cdot7$
$d_2 = 0\cdot6$	
	$r_3 = -\,37\cdot9$
$d_3 = 0\cdot6$	
	$r_4 = +\,19\cdot3$
$d_4 = 1\cdot3$	
	$r_5 = -\,31\cdot2$
$d_5 = 0\cdot6$	
	$r_6 = +\,31\cdot2$
$d_6 = 0\cdot8$	

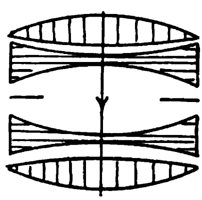
Symmetrical.

Kind.	n_D	$n_{G'}$
D.B.C.	1·6155	1·6297
Air	1·0	1·0
C.	1·5221	1·5352
D.B.C.	1·6155	1·6297
C.	1·5221	1·5352

Symmetrical.

57. Artar (*Zschokke*, 1904).

K, 276.



$f_D = 100.$

Separations.	Radii.
	$r_1 = +\,21\cdot3$
$d_1 = 2\cdot0$	
	$r_2 = -\,36\cdot9$
$d_2 = 0\cdot4$	
	$r_3 = -\,29\cdot6$
$d_3 = 0\cdot9$	
	$r_4 = +\,22\cdot0$
$d_4 = 6\cdot4$	

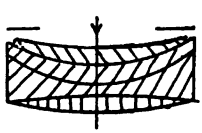
Symmetrical.

Kind.	n_D	$n_{G'}$
D.B.C.	1·6090	1·6222
Air	1·0	1·0
L.F.	1·5254	1·5386

Symmetrical.

58, Pantar (*Hoegh*).

G.P. 171369. 1904. K, 276.



$f_D = 100.$

Separations.	Radii.
	$r_1 = -\,14\cdot2$
$d_1 = 1\cdot1$	
	$r_2 = -\,\ \ 8\cdot9$
$d_2 = 1\cdot3$	
	$r_3 = -\,\ \ 6\cdot6$
$d_3 = 0\cdot6$	
	$r_4 = +\,35\cdot1$
$d_4 = 1\cdot6$	
	$r_5 = -\,15\cdot1$

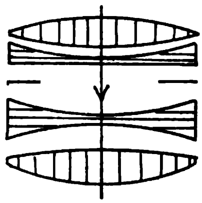
Kind.	n_D	$n_{G'}$
F.	1·6253	1·6465
C.	1·4643	1·4732
C.	1·5414	1·5563
D.B.C.	1·6210	1·6349

59. Celor $f/3{\cdot}5$ (*Hoegh*).

G.P. 202083. K, 277.

$f_D = 100.$

Separations.	Radii.	Glass. Kind.	Constants. n_D	n_G
	$r_1 = +\ 43{\cdot}5$			
$d_1 = 5{\cdot}2$		D.B.C.	1·6141	1·6280
	$r_2 = -\ 70{\cdot}4$			
$d_2 = 1{\cdot}6$		Air	1·0	1·0
	$r_3 = -\ 55{\cdot}9$			
$d_3 = 2{\cdot}1$		L.F.	1·6051	1·6260
	$r_4 = +\ 163{\cdot}9$			
$d_4 = 12{\cdot}0$		Air	1·0	1·0
	$r_5 = -\ 41{\cdot}7$			
$d_5 = 1{\cdot}8$		L.F.	1·5513	1·5672
	$r_6 = +\ 41{\cdot}7$			
$d_6 = 3{\cdot}6$		Air	1·0	1·0
	$r_7 = +\ 78{\cdot}1$			
$d_7 = 5{\cdot}7$		C.	1·6141	1·6280
	$r_8 = -\ 34{\cdot}4$			



(c) By Steinheil.

60. Orthostigmat. 1st Type. (*Steinheil*, 1903.) Patent not accepted. E, 183.

61. Orthostigmat. 2nd Type.

Steinheil (*Beck in England*).

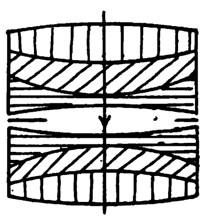
G.P. 88503. 1893.

C, 493, D, 81, E, 186.

$f_D = 100.$

Separations.	Radii.	Kind.	n_D	$n_{G'}$
	$r_1 = +\ 29{\cdot}23$			
$d_1 = 3{\cdot}94$		D.B.C.	1·60971	1·62522
	$r_2 = -\ 22{\cdot}74$			
$d_2 = 3{\cdot}94$		C.	1·53850	1·55147
	$r_3 = -\ 12{\cdot}70$			
$d_3 = 0{\cdot}88$		B.L.F.	1·57172	1·58772
	$r_4 = +\ 31{\cdot}71$			
$d_4 = 9{\cdot}0$				

Symmetrical.

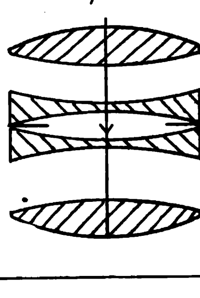


Symmetrical.

62. Unofocal (*Steinheil*, 1901).

E, 195.

Symmetrical.

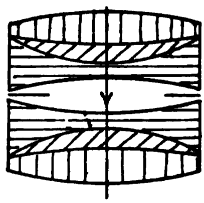


Symmetrical.

(*d*) By Voigtlaender.

63. Collinear (*Scheffer*). G.P. 88505. 1894. D, 78, E, 187. K, 280.

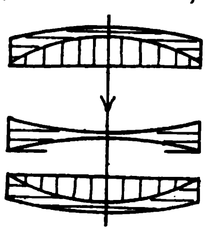
64. (Apochromat Collinear) (*Harting*). C, 422, K, 280.



Separations.	Radii.	Kind.	n_D	$n_{G'}$
$f_D = 100.$				
	$r_1 = +27\cdot104$			
$d_1 = 1\cdot965$		D.B.C.	1·6129	1·62682
	$r_2 = -33\cdot588$			
$d_2 = 5\cdot119$		C.	1·5152	1·52768
	$r_3 = -12\cdot434$			
$d_3 = 0\cdot594$		B.S.F.	1·5434	1·55738
	$r_4 = +26\cdot989$			
$d_4 = 7\cdot260$				
Symmetrical.		Symmetrical.		

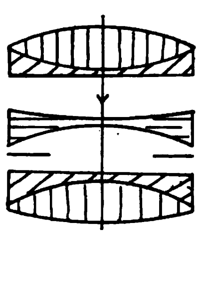
(The header over Kind. and Constants. reads "Glass.")

65. Heliar (*Harting*). G.P. 124934. 1902. K, 280.



Separations	Radii	Kind	n_D	ν
$f_D = 18.$				
	$r_1 = +\ 67\cdot9$			
$d_1 = 2\cdot8$		L.F.	1·549	45·9
	$r_2 = +\ 32\cdot5$			
$d_2 = 8\cdot5$		D.B.C.	1·614	56·2
	$r_3 = +197\cdot1$			
$d_3 = 16\cdot9$		Air	1·0	
	$r_4 = -\ 66\cdot4$			
$d_4 = 1\cdot1$		F.	1·537	51·6
	$r_5 = +\ 57\cdot1$			
$d_5 = 9\cdot0$		Air	1·0	
	$r_6 = +180\cdot3$			
$d_6 = 8\cdot7$		D.B.C.	1·614	56·2
	$r_7 = -\ 32\cdot5$			
$d_7 = 1\cdot1$		L.F.	1·581	41·9
	$r_8 = -\ 68\cdot3$			

66. Dynar (*Harting*). G.P. 143889. 1902. K, 280.

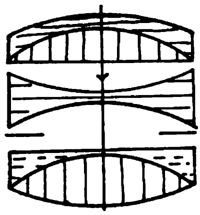


Separations	Radii	Kind	n_D	ν
$f_D = 12.$				
	$r_1 = +\ 31\cdot9$			
$d_1 = 4\cdot7$		D.B.C.	1·613	56·5
	$r_2 = -\ 32\cdot1$			
$d_2 = 1\cdot0$		C.	1·569	51·3
	$r_3 = \infty$			
$d_3 = 2\cdot9$		Air	1·0	
	$r_4 = -\ 66\cdot3$			
$d_4 = 0\cdot5$		L.F.	1·551	45·2
	$r_5 = +\ 30\cdot1$			
$d_5 = 8\cdot1$		Air	1·0	
	$r_6 = -164\cdot4$			
$d_6 = 0\cdot5$		C.	1·569	51·3
	$r_7 = +\ 25\cdot7$			
$d_7 = 4\cdot1$		D.B.C.	1·613	56·5
	$r_8 = -\ 42\cdot6$			

67. Oxyn (*Harting*).

G.P. 154910. 1903.

K, 280.



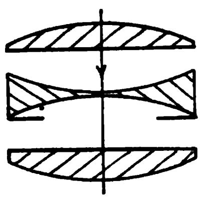
$f_D = 100.$

Separations.	Radii.	Glass. Kind.	Constants. n_D	$n_{G'}$
	$r_1 = + 34 \cdot 99$			
$d_1 = 0 \cdot 67$		L.F.	1·54890	1·56547
	$r_2 = + 20 \cdot 83$			
$d_2 = 4 \cdot 17$		D.B.C.	1·61340	1·62736
	$r_3 = \infty$			
$d_3 = 5 \cdot 83$		Air	1·0	1·0
	$r_4 = - 36 \cdot 67$			
$d_4 = 0 \cdot 42$		L.F.	1·55019	1·56597
	$r_5 = + 28 \cdot 25$			
$d_5 = 3 \cdot 17$		Air	1·0	1·0
	$r_6 = - 266 \cdot 06$			
$d_6 = 0 \cdot 42$		Si. Glass	1·53780	1·55143
	$r_7 = + 19 \cdot 16$			
$d_7 = 6 \cdot 67$		D.B.C.	1·61340	1·62736
	$r_8 = - 30 \cdot 19$			

(e) By H. D. Taylor (Cooke Lenses).

68. Rapid Portrait.

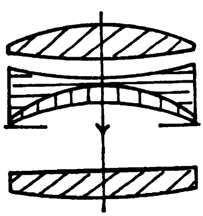
B.P. 22607. 1893.
D. 92, G. 187.



Separations.	Radii.	Kind.	n_D	$n_{G'} - n_D$
	$r_1 = + 26 \cdot 36$			
$d_1 = 5 \cdot 9$		C.	1·5108	·01037
	$r_2 = - 150 \cdot 7$			
$d_2 = 10 \cdot 9$		Air	1·0	
	$r_3 = - 29 \cdot 8$			
$d_3 = 0 \cdot 2$		F.	1·6042	·02086
	$r_4 = + 24 \cdot 15$			
$d_4 = 12 \cdot 5$		Air	1·0	
	$r_5 = + 150 \cdot 7$			
$d_5 = 5 \cdot 9$		C.	1·5108	·01037
	$r_6 = - 26 \cdot 36$			

69. Cooke, Ser. IIIa.

See 68.

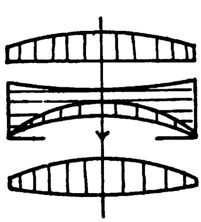


Separations.	Radii.	Kind.	n_D	$n_{G'} - n_D$
	$r_1 = + 21 \cdot 58$			
$d_1 = 6 \cdot 03$		C.	1·5101	·0101
	$r_2 = - 46 \cdot 55$			
$d_2 = 0 \cdot 8$		Air	1·0	
	$r_3 = - 34 \cdot 72$			
$d_3 = 0 \cdot 44$		Si-Glass	1·5365	·01348
	$r_4 = + 11 \cdot 5$			
$d_4 = 2 \cdot 18$		C.	1·6110	·01386
	$r_5 = + 19 \cdot 1$			
$d_5 = 9 \cdot 0$		Air	1·0	
	$r_6 = + 126 \cdot 5$			
$d_6 = 3 \cdot 93$		C.	1·5101	·0101
	$r_7 = + 58 \cdot 4$			

70. Cooke, Ser. II.

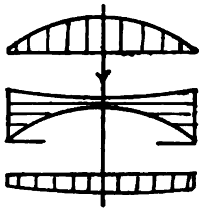
B.P. 15107. 1895.

See 68.



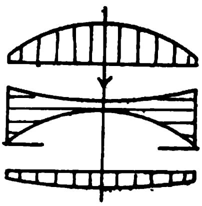
Separations.	Radii.	Kind.	n_D	$n_{G'} - n_D$
	$r_1 = + 29 \cdot 4$			
$d_1 = 6 \cdot 35$		D.B.C.	1·5751	·01286
	$r_2 = - 116 \cdot 1$			
$d_2 = 5 \cdot 0$		Air	1·0	
	$r_3 = - 46 \cdot 37$			
$d_3 = 0 \cdot 65$		Si. F.	1·5482	·01560
	$r_4 = + 13 \cdot 65$			
$d_4 = 3 \cdot 3$		C.	1·6114	·01389
	$r_5 = + 23 \cdot 85$			
$d_5 = 17 \cdot 06$		Air	1·0	
	$r_6 = + 14 \cdot 27$			
$d_6 = 3 \cdot 5$		D.B.C.	1·5751	·01286
	$r_7 = - 55 \cdot 7$			

71. Cooke, Ser. IIIb.
See 70.



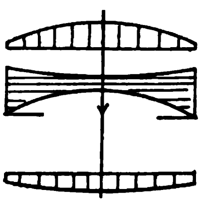
Separations.	Radii.	Glass. Kind.	Constants. n_D'	$n_G'-n_D$
	$r_1 = +\ 17{\cdot}0$			
$d_1 = 4{\cdot}0$		D.B.C.	1·6114	·01389
	$r_2 = -\ 94{\cdot}5$			
$d_2 = 0{\cdot}8$		Air	1·0	
	$r_3 = -\ 56{\cdot}0$			
$d_3 = 0{\cdot}7$		L.F.	1·5679	·01707
	$r_4 = +\ 15{\cdot}9$			
$d_4 = 11{\cdot}2$		Air	1·0	
	$r_5 = +\ 362{\cdot}3$			
$d_5 = 2{\cdot}83$		D.B.C.	1·6114	·01389
	$r_6 = -\ 77{\cdot}0$			

72. Cooke, Ser. IV.
See 70.



Separations.	Radii.	Kind	n_D	$n_G'-n_D$
	$r_1 = +\ 19{\cdot}44$			
$d_1 = 4{\cdot}29$		D.B.C.	1·6110	·01386
	$r_2 = -\ 128{\cdot}3$			
$d_2 = 1{\cdot}63$		Air	1·0	
	$r_3 = +\ 57{\cdot}85$			
$d_3 = 0{\cdot}73$		L.F.	1·5754	·01810
	$r_4 = -\ 18{\cdot}19$			
$d_4 = 12{\cdot}9$		Air	1·0	
	$r_5 = +\ 311{\cdot}3$			
$d_5 = 3{\cdot}03$		Crown	1·6110	·01386
	$r_6 = -\ 66{\cdot}4$			

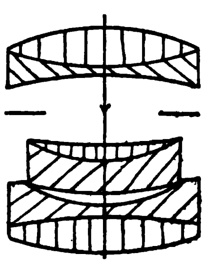
73. Cooke, Ser. V. *See* 70.
Also C, 490.



Separations.	Radii.	Kind	n_D	$n_G'-n_D$
	$r_1 = +\ 14{\cdot}57$			
$d_1 = 29{\cdot}9$		D.B.C.	1·6114	·01389
	$r_2 = -\ 101{\cdot}3$			
$d_2 = 0{\cdot}38$		Air	1·0	
	$r_3 = -\ 55{\cdot}93$			
$d_3 = 4{\cdot}6$		L.F.	1·5482	·01559
	$r_4 = +\ 13{\cdot}27$			
$d_4 = 8{\cdot}95$		Air	1·0	
	$r_5 = +\ 101{\cdot}2$			
$d_5 = 1{\cdot}83$		D.B.C.	1·6114	·01389
	$r_6 = -\ 69{\cdot}75$			

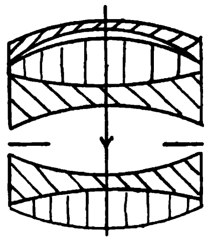
(*f*) By J. H. Dallmeyer. Ltd. (Willesden, London).

74. Stigmatic, Ser. I
(*T. R. Dallmeyer and Aldis*).
B.P. 16640. 1895. D, 86.



Separations.	Radii.	Kind	n_D	ν
	$r_1 = +\ 25{\cdot}0$			
$d_1 = 7{\cdot}0$		D.B.C.	1·5726	57·5
	$r_2 = -\ 36{\cdot}39$			
$d_2 = 1{\cdot}67$		F.	1·3738	41·4
	$r_3 = +\ 40{\cdot}89$			
$d_3 = 12{\cdot}0$		Air	1·0	
	$r_4 = -\ 58{\cdot}12$			
$d_4 = 4{\cdot}44$		D.B.C.	1·5726	57·5
	$r_5 = -\ 14{\cdot}44$			
$d_5 = 1{\cdot}67$		C.	1·5151	56·6
	$r_6 = -\ 32{\cdot}11$			
$d_6 = 2{\cdot}22$		Air	1·0	
	$r_7 = -\ 17{\cdot}22$			
$d_7 = 1{\cdot}67$		C.	1·5151	56·6
	$r_8 = +\ 53{\cdot}32$			
$d_8 = 5{\cdot}0$		D.B.C.	1·5726	57·5
	$r_9 = -\ 33{\cdot}47$			

75. Stigmatic, Ser. II
(Convertible). *See* 74.

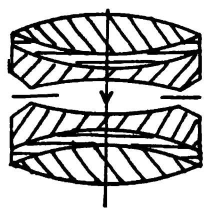


Separations.	Radii.	Glass. Kind.	Constants. n_D	ν
	$r_1 = +19 \cdot 96$			
$d_1 = 0 \cdot 8$		F.	1·5151	56·6
	$r_2 = +14 \cdot 23$			
$d_2 = 1 \cdot 67$		Air	1·0	
	$r_3 = +17 \cdot 65$			
$d_3 = 4 \cdot 44$		D.B.C.	1·5726	57·5
	$r_4 = -33 \cdot 33$			
$d_4 = 3 \cdot 61$		F.	1·5368	
	$r_5 = +22 \cdot 83$			
$d_5 = 6 \cdot 11$		Air	1·0	
	$r_6 = -22 \cdot 22$			
$d_6 = 0 \cdot 83$		F.	1·5368	
	$r_7 = +33 \cdot 33$			
$d_7 = 3 \cdot 61$		D.B.C.	1·5726	57·5
	$r_8 = -19 \cdot 24$			

76. Stigmatic (non-convertible), **Ser. III.** *See* 74.
77. Serrac.

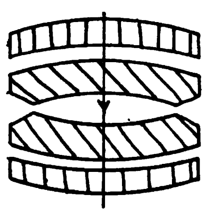
(*g*) By Ross.

78. Universal Symmetric Anastigmat. D, 86.



Not patented.

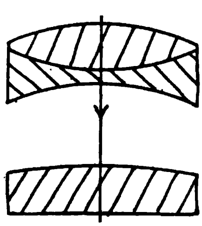
79. Homocentric. 1902 ?



80. Compound Homocentric. 1907 ?
81. Xpres.

(*h*) By Aldis.

82. Aldis, Ser. II (*H. L. Aldis*).
B.P. 10016. 1902.

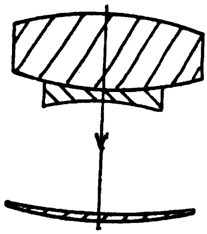


Data not Published.

Data not Published.

Data not Published.

83. Aldis, Ser. III. *See* 82.



Data not Published.

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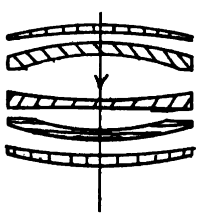
84. *Also* Duo 1907? Trio 1908? Oxys 1908?

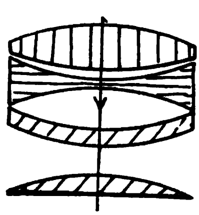
(i) By R. & J. Beck (Cornhill, London).

85. Isostigmar.
B.P. 27180. 1906.



Separations.	Radii.	Glass. Kind.	Constants. n_D	ν
	$r_1 = +\ 2{\cdot}19$			
$d_1 = 0{\cdot}18$		D.B.C.	1·6065	58·1
	$r_2 = +\ 7{\cdot}4$			
$d_2 = 0{\cdot}236$		Air	1·0	
	$r_3 = +\ 2{\cdot}69$			
$d_3 = 0{\cdot}097$		L.F.	1·6193	37·4
	$r_4 = +\ 1{\cdot}62$			
$d_4 = 0{\cdot}569$		Air	1·0	
	$r_5 = -\ 29{\cdot}7$			
$d_5 = 0{\cdot}076$		C.	1·5193	58·7
	$r_6 = +\ 35{\cdot}3$			
$d_6 = 0{\cdot}118$		Air	1·0	
	$r_7 = -\ 1{\cdot}59$			
$d_7 = 0{\cdot}139$		L.F.	1·5703	41·1
	$r_8 = -\ 2{\cdot}708$			
$d_8 = 0{\cdot}215$		Air	1·0	
	$r_9 = -\ 7{\cdot}4$			
$d_9 = 0{\cdot}18$		D.B.C.	1·6065	58·1
	$r_{10} = -\ 2{\cdot}19$			

86. Neostigmar (convertible).
B.P. 22157.* 1910.



Separations.	Radii.	Kind.	n_D	ν
	$r_1 = +\ 1{\cdot}78$			
$d_1 = 0{\cdot}43$		D.B.C.	1·6128	58 3
	$r_2 = -\ 10{\cdot}03$			
$d_2 = 0{\cdot}03$		Air	1·0	
	$r_3 = -\ 5{\cdot}9$			
$d_3 = 0{\cdot}08$		L.F.	1·5580	45·5
	$r_4 = +\ 1{\cdot}64$			
$d_4 = 0{\cdot}41$		Air	1·0	
	$r_5 = -\ 5{\cdot}3$			
$d_5 = 0{\cdot}17$		C.	1·5289	58·9
	$r_6 = -\ 3{\cdot}9$			
$d_6 = 0{\cdot}33$		Air	1·0	
	$r_7 = +\ 9{\cdot}3$			
$d_7 = 0{\cdot}25$		C.	1·5289	58·9
	$r_8 = \infty$			

* This Patent includes also a number of Triple objectives.

CHAPTER XV.

Ophthalmic Optics.

148. Spectacle Lenses. General Remarks.

The use and action of spectacle lenses have already been mentioned when dealing with the dioptrics of the eye in Chapter VI. Any centred system of spherical refracting surfaces of suitable power, if it change the refraction of the eye in the desired manner, may be used as a spectacle glass. Since spectacle lenses for general use must be small in weight and reasonable in price, they are usually made in the form of thin lenses. These may be either bi-convex, in which case (*see* § 47) they exhibit strong astigmatism and a small amount of spherical aberration; or they may be meniscus in form with the concave surface towards the eye; in this case the astigmatism appears diminished, but the spherical aberration is increased. These two faults are first noticeable only when the power of the lenses becomes comparatively high (about 10 *Dp.*). The meniscus lenses—also called **periscopic**—were suggested by Wollaston. It is possible to correct them completely for astigmatism for a definite inclination of the rays. The subject has been dealt with by Tscherning, Ostwald and Percival. These lenses have, however, not attained much practical importance since comparatively little is gained by eliminating astigmatism alone, without correspondingly diminishing the other errors of image formation. For example, distortion, which is a result of the difference in focal length between the central and edge rays, and chromatism are not eliminated. Recently Zeiss, by employing, of course, additional optical means, have produced spectacle lenses satisfying these requirements to an appreciable extent; in one case the improvement consists in the use of two cemented lenses, and in the other by employing a surface which is not spherical. Following Gullstrand, the point of rotation of the human eye is taken as the point of intersection of the principal rays, along which the correction is applied; this point lies about 3 cm. behind the vertex of the spectacle lens.

149. Achromatic Spectacle Lenses of Zeiss.

Strong convex achromatic spectacle lenses, as required for aphakic eyes, are made by Zeiss*; in these lenses

* German Patent No. 219895.

astigmatism also is eliminated. The corrections, as mentioned above, are referred to the point of rotation of the eye, which is about 3 cm. behind the spectacle lens.

Two constructions are represented in Figs. 202 and 203. The red and blue component rays which correspond, in the object space, to the principal rays chosen for achromatic correction in the image space, are inclined to the axis at angles w_C and w_F respectively. In the case of very distant objects, the distances from the axis (h_C and h_F) of the points in which these components intersect the object plane are given instead of the angles w_C and w_F. The angle of inclination w of three principal rays in the object space, and the intercepts s_s' and s_t' of the sagittal and meridional rays corresponding to them—that is, the distances in the image space from the points of emergence of the bundles to their intersection planes—are also given.

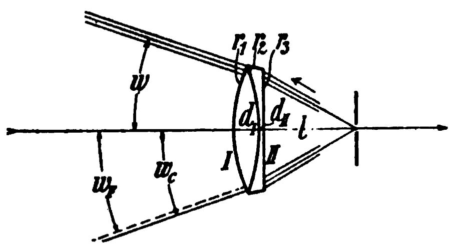


FIG. 202.

Focal Length = 91·02 mm. Power = 11 *Dp.* Very distant objects.

$n_{DI} = 1{\cdot}51633$ $\nu_I = 64{\cdot}1.$ $n_{DII} = 1{\cdot}60291.$ $\nu_{II} = 38{\cdot}1.$

$r_1 = 40{\cdot}92$ mm. $r_2 = 50{\cdot}0$ mm. $r_3 = \infty.$

$d_I = 7{\cdot}0$ mm. $d_{II} = 1{\cdot}0$ mm. $l = 25{\cdot}0$ mm.

$w_C = 19^\circ{\cdot}353.$ $w_F = 19^\circ{\cdot}352.$

$w =$	0°.	13°·481.	19°·347.
$s_s' =$	85·09 mm.	87·84 mm.	92·10 mm.
$s_t' =$	85·9 mm.	86·13 mm.	94·45 mm.

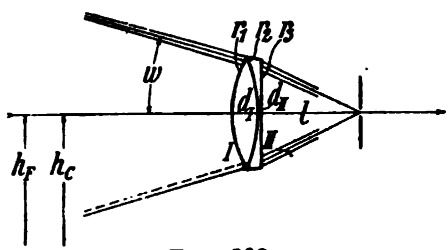


FIG. 203.

Focal Length = 69·04 mm. Power = 14·5 *Dp*. Object plane perpendicular to the axis at a distance of 315 mm. from the lens.

$n_{DI} = 1{\cdot}51687$. $\nu_I = 64{\cdot}1$. $n_{DII} = 1{\cdot}59633$. $\nu_{II} = 39{\cdot}6$.

$r_1 = 31{\cdot}93$ mm. $r_2 = 43{\cdot}21$ mm. $r_3 = \infty$.

$d_I = 7{\cdot}0$ mm. $d_{II} = 0{\cdot}5$ mm. $l = 25{\cdot}0$ mm.

$h_C = 109{\cdot}34$ mm. $h_F = 109{\cdot}13$ mm.

$w =$	0°.	$11^\circ{\cdot}370$.	$16^\circ{\cdot}343$.
$s_s' =$	82·95 mm.	84·81 mm.	88·34 mm.
$s_t' =$	82·85 mm.	82·57 mm.	92·06 mm.

150. Deformed Menisci of Zeiss.

By deforming one of the surfaces, Zeiss have been able to reduce considerably the difference between the focal lengths for the peripheral and central rays, thereby practically eliminating distortion and the difference in magnification between the paraxial and oblique bundles. As before the correction is referred to the rotation point of the eye. Figs. 204 and 205 illustrate the forms of two such lenses.

The first is a distance lens for a strongly myopic eye, and the second a reading lens for a strongly hypermetropic or an aphakic eye.

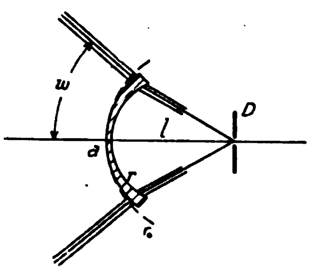


FIG. 204.

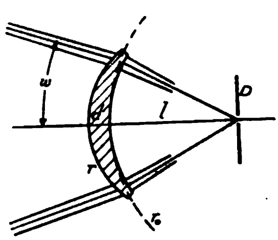


FIG. 205.

The diverging meniscus (Fig. 204) consists of glass of refractive index 1·52, the focal length being 100 mm. and the power consequently 10 *Dp*. The objects are at a great distance and it is also assumed that a narrow back stop *D* exists at a distance $l = 30$ mm. from the vertex of the second surface. The radius r of the second surface is 15·8 mm., the thickness at the vertex $d = 0{\cdot}5$ mm. The first surface is slightly deformed, its radius r_0 in the neighbourhood of the vertex being 23·0 mm. Over a diameter of 32 mm. the

distance from the edge of the deformed surface to the edge of the spherical surface of radius 23 mm. (shown in dotted lines) is only $\frac{1}{4}$ mm.

From the following data it will be seen in what measure the correction of astigmatism and distortion is attained. Firstly, with regard to astigmatism, the intercepts s_s' of the sagittal and s_t' of the meridional components of the emerging bundles are given for several inclinations w of the incident bundles—*i.e.*, the distances of the points of emergence from the planes of intersection. Secondly, with regard to distortion, which without deformation is strongly barrel shaped, the quantity V, called the **magnification ratio**, is given. In this case, this quantity V is the ratio of the angular distance of the object point to the linear distance of the image point from the optical axis.

$w =$	0°.	$26^\circ 265$.	$37^\circ 201$.
$s_s' =$	99·26 mm.	109·29 mm.	101·82 mm.
$s_t' =$	99·26 mm.	97·88 mm.	102·73 mm.
$V =$	100·00.	99·01.	98·31.

The converging meniscus in Fig. 205 is made of glass of refractive index 1·655. Its focal length is 72·59 mm. and therefore its power is 13·8 Dp. It is assumed that the object lies in a plane perpendicular to the axis and 330 mm. from the vertex of the first surface, and that a narrow stop D is placed at a distance l = 30 mm. from the vertex of the second surface. The radius r of the front surface is 21·30 mm. The thickness at the vertex is 5·0 mm. The back surface is deformed. The radius r_0 of the spherical surface of the same curvature as that of the deformed surface at the vertex is 35·0 mm., whilst the edge of the deformed surface over a diameter of 34 mm., is distant $\frac{1}{4}$ mm. from the edge of this spherical surface. The following table gives the same quantities as in the previous example, except that in this case V is the ratio of two distances and may therefore be expressed as a percentage. Without the use of the deformed surface the distortion would be markedly cushion in shape.

$w =$	0°.	$10^\circ 596$.	$16^\circ 123$.
$s_s' =$	86·66 mm.	88·27 mm.	91·36 mm.
$s_t' =$	86·66 mm.	87·19 mm.	93·78 mm.
$V =$	100·00%.	100·57%.	100·56%.

According to Zeiss*, a weak meniscus is sufficiently astigmatically corrected for distance when the radius of the corrected surface in millimetres can be represented as a fraction with numerator 1000 and a denominator consisting of two terms, of which one is the power of the lens in dioptres—positive in the case of a converging lens and negative for a diverging lens—whilst the other term has a positive value between 10 and 25.

151. The Double Meniscus of Zeiss.

Fig. 206 shows a corrected system consisting of two menisci, similar to the verant systems (*see* § 75), in which, in addition, astigmatism is eliminated.†

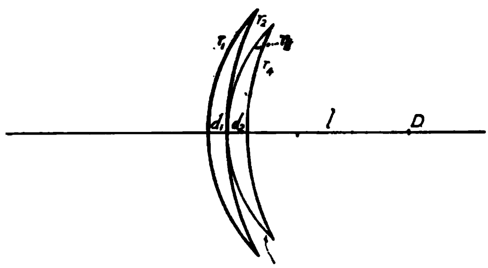


FIG. 206.

The rotation point of the eye, D, is again taken as the point of intersection of the principal rays, its distance from the vertex of the second lens being represented by l. Below are given the constructional data for three different forms. The two menisci are in contact on the axis.

In System I the first lens is bi-convex.	I. The system to be used as a magnifying glass.	II. The system corrects an aphakic eye for distance.	III. The system corrects an aphakic eye for a distance of 300 mm.
	All measurements in millimetres.		
Refractive Indices of the two menisci.	1·52	1·61	1·61
Power in Dioptres	22·4	11	14·3
r_1	52·9	43	46·04
r_2	−300	68·2	95·64
r_3	43·7	29·2	30·5
r_4	800	38·6	46·3
d_1	4	3	3·5
d_2	4	2	2·8
l	29	26	26·0

* *See* German Patent No. 217963.

† For a description of this system *see* German Patent No. 213868.

152. Telescopic Spectacle Lenses.

As long ago as the 18th Century* it was proposed to replace simple spectacle lenses by a combination on the Galilean telescope principle, consisting of two separated lenses, the front one being positive and the hinder one negative. In strong myopia, combined with weakness of vision, combinations of this kind are decidedly useful. These combinations have not, up to the present time, found their way into practice, principally because of their great weight and insufficient corrections; but recently interest in systems of this kind has been revived.† The author, in

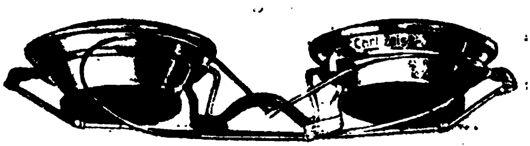


FIG. 207.

conjunction with Dr. Plehn, has evolved a method by which, given the extent of the weakness of vision of a person, measured in a certain way, the two components of such an optical combination may be calculated, thereby compensating this weakness of vision. Further, v. Rohr‡ has given a complete theory of this combination. The lenses made by Zeiss according to this theory are astigmatically corrected and have, at least for weak magnifications up to about 1·3, a sufficiently large field of view and a comparatively small weight. Lenses for a myope of 16 *Dp.* weigh only 35 grm. and are made in whole dioptres from 10 to 20. Fig. 207 illustrates telescopic spectacle-lenses of this kind

153. Ophthalmoscopes. General Remarks.

In its widest sense the term ophthalmoscope is applied to any optical arrangement by means of which the interior of the eye, and particularly the retina, may be examined. Such arrangements have very different constructions and receive various names such as retinoscopes, ophthalmoscopes, fundoscopes. It may be stated without exaggeration that since the first ophthalmoscope was constructed in 1852 by Helmholtz, over a hundred different constructions of

* *See* British Patent No. 1515 of 1785 by Dixon.
† *See* "Der Mechaniker," 1910, Pts. 4–6.
‡ *See* "Zur Theorie der Fernrohrbrille" in v. Graefes Archiv. für Ophthalmologie. Vol. LXXV, Part 3.

apparatus of this kind have been described in various scientific papers and patent specifications. Very few of these instruments have, however, come into general use.* Apart from purely mechanical improvements, the numerous variations of this instrument are no better than the ophthalmoscope due to Prof. Ruete (having a concave mirror with a central hole) which was constructed only a few months after the invention of Helmholtz. It is only within the last twenty years or so that, along with the general progress of technical optics, this important instrument has been improved. Notwithstanding the numerous different forms of existing ophthalmoscopes, the principles on which they rest are few and will be described below.

Every ophthalmoscope has two essential parts: an illuminating apparatus and an observing apparatus. The function of the former is to make the background of the eye self-luminous by projecting light on to it. There can be distinguished four different methods of illumination.

1. Between the eye A_1 of the patient and the eye A_2 of the physician a plane sheet of glass is interposed obliquely relative to the optical axis. As will be seen in Fig. 208, part of the light from the source L is reflected from the glass plate S, after which it travels parallel to the optical axis, enters the eye A_1 through the pupil EJ, and is brought to a focus at the point P on the retina, assuming

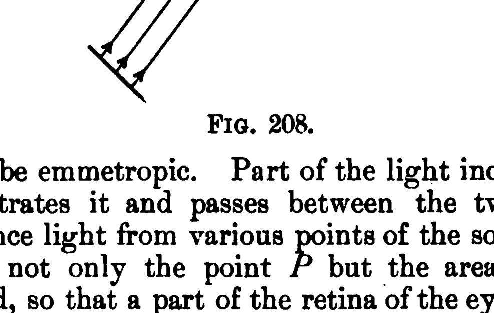


FIG. 208.

the eye to be emmetropic. Part of the light incident on the plate penetrates it and passes between the two eyes and away. Since light from various points of the source falls on the plate, not only the point P but the area around it is illuminated, so that a part of the retina of the eye A_1 is made to a certain extent self-luminous. The light from this portion

* In the work "Der Augenspiegel" by Adolf Zander, Leipzig and Heidelberg 1859, thirty-four different ophthalmoscopes are described.

returns through the pupil EJ; a portion is reflected back to the source, and the remainder, passing through the plate, enters the eye A_2 of the physician, in which it unites at the sharp image point P' if this eye be emmetropic. By means of an arrangement of this kind. devised as already remarked, by Helmholtz, the background of the eye was first systematically explored.

2. Only a short time after the introduction of this apparatus by Helmholtz, a second arrangement for projecting light into the pupil of the eye was suggested by Ruete and this is the form most frequently used at the present time. A concave mirror S (Fig. 209) with a hole W bored through its centre is placed between the two eyes so that the circular

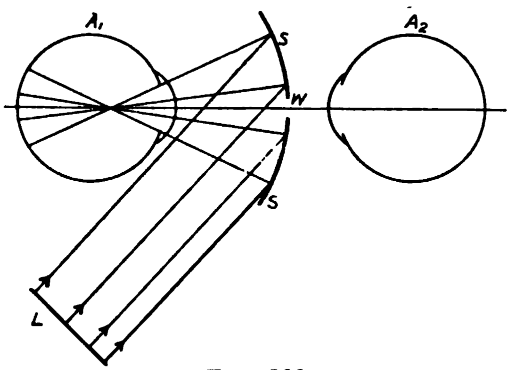


FIG. 209.

hole is perpendicular (or very nearly so) to the common optical axis of the eyes. The whole of the light from the source L_1 incident on the concave mirror is reflected through the pupil of the eye A_1 and illuminates the background. The physician's eye A_2, looking through the central aperture W, receives the rays returning from A_1.

3. In the arrangements just described the light entering the eye of the patient and that which returns and enters the eye of the physician traverse the same path, but other arrangements have been devised in which these paths are separated. One tube is provided through which observations can be made, whilst a second one, usually called the light tube, is separated from it and guides the light into the eye of the patient. In this case the source of light is usually an electric lamp, whose effect may be increased by lenses placed in front of it or by a curved mirror placed behind. The observing tube is likewise provided with several lenses acting as a telescope. An arrangement of this kind was first suggested by Ulrich. It should be noticed however, that those parts of the cornea

through which the light passes in and out are not completely separated from one another, but overlap, and consequently, as will be seen more fully below, undesirable reflections are set up.

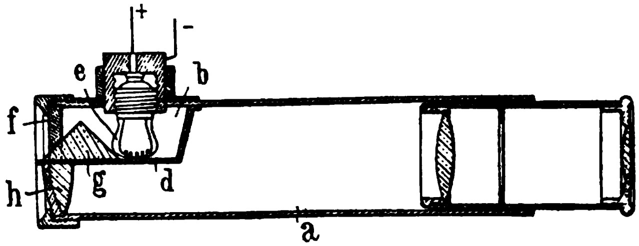


FIG. 210.

One of the latest ophthalmoscopes of this kind is that of Baum. It consists (Fig. 210) of a tube *a*, containing an optical arrangement similar to that of a telescope, one half of the objective having been removed. The partition *d* encloses a chamber *b*, containing a prism *g* and a lamp *e*. The forward end of this chamber is closed by a diaphragm *f* as far as the surface of the prism, leaving only a narrow slit for the light to pass through. The light from the lamp passing through this slit shows up quite straight and sharp along the side of the half objective. This light enters through one half of the pupil and illuminates the background of the eye, so that the image as observed through the eye-piece and the objective *h* is comparatively free from reflections.

4. A few years after the construction of the ophthalmoscope by Helmholtz, attempts were made to separate completely that part of the pupil through which the light enters, from

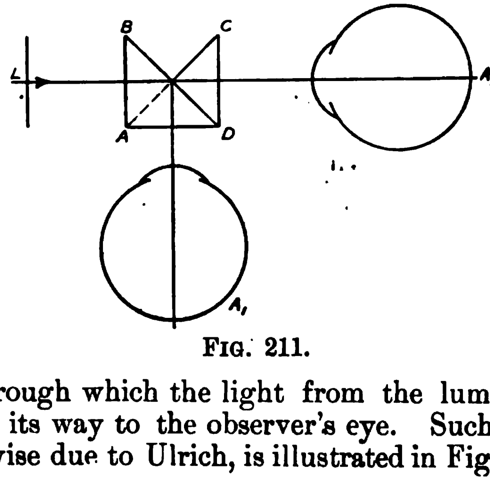


FIG. 211.

the part through which the light from the luminous retina emerges on its way to the observer's eye. Such an arrangement, likewise due to Ulrich, is illustrated in Fig. 211. Two

glass prisms ABD and ACD, whose sections are right-angled isosceles triangles, are cemented together with their hypotenusal faces perpendicular to one another and the side AD in each coincident. The light from the source L, placed at the side, is totally reflected by the face BD of the upper prism BAD and enters the eye A_1 of the patient. The rays returning from the retina of A_1 are totally reflected at the face AC of the lower prism ACD and pass into the eye A_2 of the physician. It will be seen below, when dealing with image formation and the elimination of reflections, why this arrangement did not meet with the success anticipated.

5. Besides these four methods of illuminating the background of the eye, experiments have been made to cause light to penetrate into the interior of the eye through the skin surrounding it by employing very bright sources of light.

The arrangements for observing the background of the eye have also undergone many changes. In some cases, no optical apparatus whatever is required; in others, single lenses are used. On the other hand somewhat complex optical arrangements have been constructed for this purpose, the theory of which will be given later.

Of the various methods employed for the examination of the eye by means of the ophthalmoscope, there are two which are particularly important and indispensable to the eye specialist. They are

1. Erect image examination.
2. Inverted image examination.

154. Erect Image Examination of the Eye.

In the following discussion, the illuminating apparatus will be completely ignored and the retina of the eye under observation will be assumed to be self-luminous. Consider first the light emanating from that point P of the self luminous retina where the latter is cut by the optical axis. For an emmetropic unaccommodated eye, incident parallel rays are brought to a focus at a point on the retina. A short sighted eye brings to a focus at this point those rays which originate at a luminous object situated at the far-point of the eye; whilst the long-sighted eye can bring to a focus at this point, only those rays which enter the pupil as a beam converging on the virtual far-point behind the eye. Applying the principle of the reversibility of light, it will be possible to consider rays starting from the point P on the retina (assumed

to be self-luminous) and passing out into space (see Figs. 212–214). In the case of the emmetropic eye (Fig. 212), these rays leave the pupil as a parallel bundle; Fig. 213

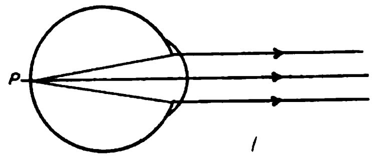


FIG. 212.

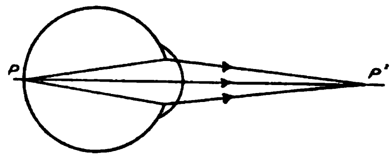


FIG. 213.

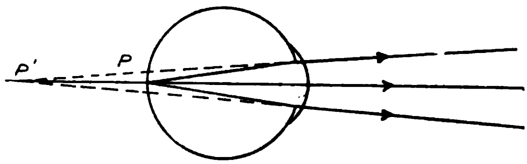


FIG. 214.

shows the case of the short-sighted eye in which the rays leaving the pupil converge on the far point P' which is conjugate to P; with the long-sighted eye (Fig. 214) the rays from P emerge from the pupil as a bundle diverging from the virtual far-point P'. If now the eye of the physician be assumed to be emmetropic and at rest, then in the first case above, the point P of the retina of the patient's eye will be seen distinctly (Fig. 215).

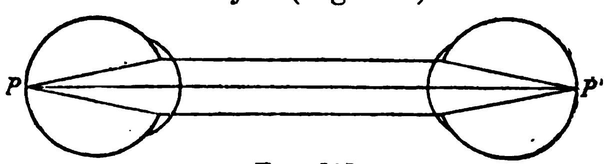


FIG. 215.

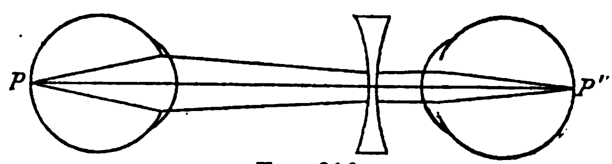


FIG. 216.

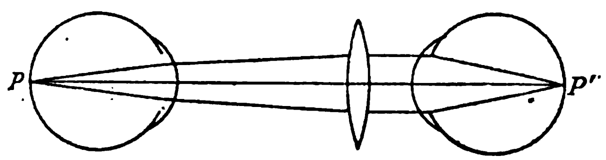


FIG. 217.

In the other two cases means must be provided to render parallel the convergent (Fig. 216) or divergent (Fig. 217) beam emerging from the patient's eye. This is effected by

the use of a diverging lens in the case of the short-sighted eye, and a converging lens in the case of a long-sighted eye—these lenses being so arranged that their respective focal points coincide with the far point of the eye under examination, as shown in Figs. 216 and 217. In this way the points P and P' are made conjugate, so that the eye of the physician receives a sharp image of the retina of the patient. It may be noticed that the power of the lens employed provides a measure of the refraction of the patient's eye referred to the vertex of the lens; hence in this way the ophthalmoscope furnishes a means of determining the refraction of the eye.

155. Inverted Image Examination of the Eye.

A short-sighted eye forms a real image of its background, this image, of course, being inverted; it may be observed directly by the physician from a distance of 25 cm.—the distance of distinct vision. The examination is then said to be carried out by inverted images. If the eye under observation is not short-sighted, the same effect may be obtained by placing in front of it a lens of suitable focal length (about 5 cm.), this lens, in combination with the eye under examination, furnishing an inverted image of the background of the latter.*

156. A Property of Luminous Surfaces. The Pupil as the Window of the Eye.

A law governing certain effects of luminous surfaces will now be demonstrated which will be found useful in explaining several phenomena.

Let L (Fig. 218) be a luminous surface in front of which is placed a stop AB. Suppose that light from L passes through this stop and through any refracting system and finally falls on a screen W. If $A'B'$ is the image of the stop $A\ B$, then it is evident that all light in the object space

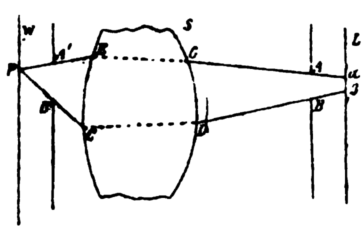


FIG. 218.

passing through $A\ B$ must, after refraction pass through the aperture $A'B'$. Suppose now that the illumination is focussed on the screen W. In order to investigate the effect

* For further information on the special conditions, magnification, field-of-view, most favourable position of lens, etc., *see* the author's "Einführung in die medizinische Optik." Leipzig, 1904, p. 220, *et seq.*

at a point P on W, resort will be made to the principle of the reversibility of light: This principle may be stated as follows: If a given ray pass through any refracting medium, it will have a definite position and direction in that medium; then if the direction of the light be reversed, the ray will re-traverse exactly its previous path in the reversed direction. In order to find the effect produced at P by the surface L, it will be observed that of the infinite number of rays from L passing through the aperture $A\ B$, a definite number will strike the point P. From the above-mentioned principle these rays may be found by taking into consideration all the rays from P which pass through the aperture $A'\ B'$. Hence according as these rays, after passing through the system S return to the luminous surface, they help in the illumination of the point P. In the present case these rays will be enclosed in the space between $P\ C\ D\ B$ and $P\ E\ G\ A$ and it will be seen that they return to the part $\alpha\beta$ of the luminous surface. Hence this portion only is to be considered with reference to the illumination of the point P.

We will now investigate the question why the pupil of the human eye appears black. It may be remarked at the outset that complete darkness by no means prevails in the interior of the human eye. A very small creature placed in the vitreous humour would receive an impression of brightness somewhat similar to that which obtains in a room provided with dark curtains. The following considerations will make this clearer. In Fig. 219, let $A\ B\ C\ D$ represent a room with an open window $E\ F$. It is a matter of common experience that if this window be observed in the daytime from a great distance, it appears completely black against the wall $B\ D$ of the house. The reason for this is evident. A small surface element e of the wall $B\ D$ receives light from the sky in all directions within an extent of 180°. A small element e' of the inner wall $A\ C$, on the contrary, receives light only from the part $G\ H$ of the sky determined by the straight lines joining e' to the points E and F and produced outwards. The diagram in Fig. 219 must be considered as representing a space of which the figure is a section. It will be seen that the element of the wall inside the room receives only a small fraction of the great quantity of light which falls on the element e of the outer

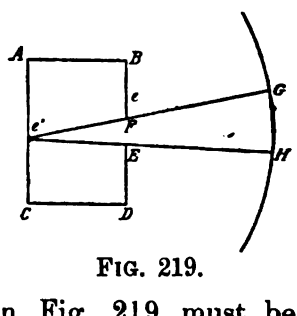


FIG. 219.

wall of the house. To a distant observer then, the window $E\ F$ will appear much less illuminated than the wall $B\ D$; and this effect will be enhanced on account of the contrast, and especially as all details in the interior disappear with increasing distance. Moreover, if the window EF be obstructed with a convex lens, the effect will remain substantially the same. In this case also, rays from a comparatively small portion only of the sky unite at each portion of the inner wall of the room.

From the rule established in the foregoing paragraph, it is only necessary to consider the rays from e' and refracted through the convex lens, in order to find that portion of the sky which takes part in the illumination. This portion is always comparatively small. The window in this case also would appear dark. It will also be necessary to take account of the light which is reflected at the first and second surfaces of the lens and sent back into space. Such reflections may be observed at any time when looking at a window. The case of the human eye is quite analagous. The window of the eye, *i.e.*, the pupil, appears dark when viewed at a distance; and, moreover, strong reflections are visible. For example, in the case of people inside a room, the reflections from the eye consist mostly of the cross-bars of the window; from people inside a ball-room the various lights in the room would be strongly reflected. In addition to the above reasons for the apparent blackness of the pupil, may be mentioned the fact that in order that one person may observe closely the pupil of the eye of another person, the former's head must be brought quite close to the pupil of the eye under observation, thus obstructing rays which might illuminate the interior of the eye. Hence the interior of the eye or the pupil actually appears a deep black. Thus, if it be desired to observe the interior of an eye, care must be taken to make the part under observation appear brighter than the surroundings, *i.e.*, light must be introduced in such a manner that the observation of the eye is not interfered with. The ophthalmoscope of Helmholtz is based on these considerations.

157. Conditions necessary for Freedom from Reflections.

In order to investigate more closely into the nature of the disturbing reflections present when examining an eye by means of an ophthalmoscope, it will be remarked in the first place that complete freedom from reflections is obtained only when that part of the pupil through which the eye is

examined is entirely separated from the portion through which pass the illuminating rays of light. Fig. 220 illustrates these conditions. The patient's eye is to the left whilst that on the right is the physician's. Both eyes are assumed to be emmetropic. Between the two eyes is placed obliquely a plane mirror with an aperture $B\ C$ through which the eye of the physician is directed. The luminous source is placed at L. Assuming that the two eyes are central, then the point P on the retina of one eye is conjugate to the point P' on that of the other eye. As explained in § 156, in order to find those rays by which P is illuminated, the rays coming from this point are traced backwards. It will be observed that two distinct systems of rays may be differentiated—a central beam $P\ G\ H$ which passes through the aperture $B\ C$ and forms a sharp image point at P' in the physician's eye; and a system of beams surrounding the first beam $P\ G\ H$. This second system, after leaving the patient's eye, falls on the mirror, from which it is reflected, according to the laws of reflection, back to the source L. This system

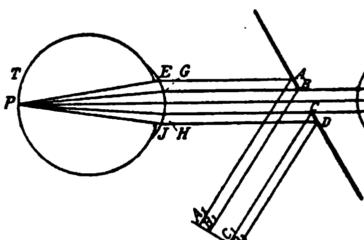


FIG. 220.

of beams in the figure is bounded on the one side by the rays $P\ E\ A\ A_1$ and $P\ G\ B\ B_1$, and on the other side by $P\ H\ C\ C_1$ and $P\ J\ D\ D_1$. Since these rays return to the source, the illumination of the point P must arise from them. It would appear at first sight as if the condition for the freedom from reflections were fulfilled, for the portion of the pupil $G\ H$ serving for the observation of the point P is quite separated from the parts $E\ G$ and $J\ H$ serving for the illumination. A closer observation, however, shows that this condition is not fulfilled in this case. Thus, consider another point T situated very close to P. On drawing a system of rays for this point similar to the system shown for P, it will be seen that some of the rays returning back towards the source pass through the portion $G\ H$ of the pupil and reflected images appear. If this construction be applied for a number of points in the neighbourhood of P, it

will be found that the portion G H is completely filled by those rays which return to the source and the complete separation between the illuminated and non-illuminated portions of the pupil is by no means obtained.

The means whereby the passage of light from the source through the part G H is prevented, will now be described. A convex lens K (Fig. 221) is placed between the mirror and the source. On tracing backwards the paths of the rays from P which are reflected at the mirror, it will be found that they are brought to a focus by the lens at its focal point F, and continue as a diverging beam to the source. Let E' G' H' J' be the plane conjugate to E G H J, and let G' H' be an opaque diaphragm so that only a ring remains for the passage of the light. This is shown in the figure as a section on the line U V.

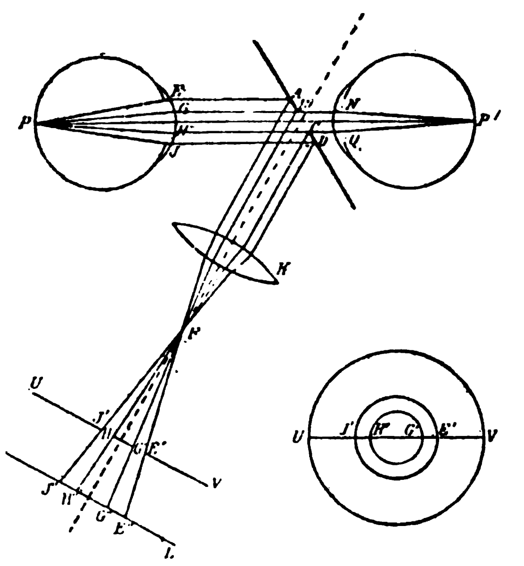


FIG. 221.

The substitution of a convex mirror in place of the plane reflector does not, in principle, alter the conditions. This insertion of another optical element merely introduces a modification in the formation of the image of the central portion G H of the pupil of the patient.

In the above arrangement, none of the rays from the source L pass through the central part G H, this latter being reserved completely for the purpose of observation by

the physician. Complete freedom from reflection cannot, however, be considered as accomplished even yet, for there is lacking a second condition to be fulfilled: namely, that the pupil of the patient's eye is conjugate to that of the physician, in other words, that $N\ Q$ is conjugate to $G\ H$. In the arrangement of Fig. 221 this is not the case, as there is no optical element between the pupils. Consequently all the rays from the patient's eye which fill the aperture $G\ H$ do not fill the aperture $N\ Q$ of the physician's eye. On the contrary, there are other beams of light from the patient's eye, which overlap the portions $E\ G$—$H\ J$ and enter the physician's eye, thus giving rise to reflections. Hence, it appears that in the examination of the eye by erect images, the condition of complete freedom from reflections is not realised. On the other hand, the interposed lens employed in the examination by inverted images, may be chosen of such a power that this condition is fulfilled. Hence the power of the lens (K in Fig. 221) must be chosen accordingly. The following conditions therefore must be fulfilled in an ophthalmoscope which is to be free from reflections:—

1. The background of the patient's eye must be conjugate to that of the physician's eye.
2. The pupil of the patient's eye must be conjugate to the pupil of the physician's eye.
3. The pupil of the patient's eye must be conjugate to the entrance pupil of the illuminating system.

Thorner, in his so-called **Stable Ophthalmoscope,** was the first to realise these three conditions successfully.*

158. More Recent Ophthalmoscope of Thorner.

The stable ophthalmoscope of Thorner, made by Schmidt and Hänsch, is an apparatus of comparatively large dimensions, and is mounted on a table specially made for it. His later ophthalmoscope† is designed for hand use. Freedom from reflections is obtained by introducing in the illuminating and observing systems two reflectors parallel to each other and arranged relatively to one another so that the above conditions are at least very approximately realised, a real image of the background of the patient's eye being formed at the distance of distinct vision of the physician's eye. The

* Thorner, "Die Theorie des Augenspiegels und die Photographie des Augenhintergrundes." Berlin, 1903.

† German Patent No. 218227.

observing aperture is placed to one side of the luminous surface. Fig. 222 represents a vertical section and Fig. 223 an end view of the instrument.

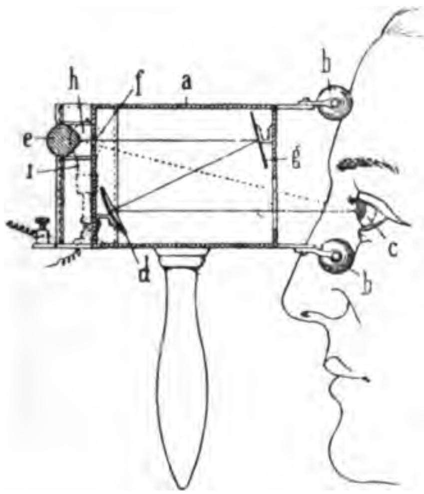


FIG. 222.

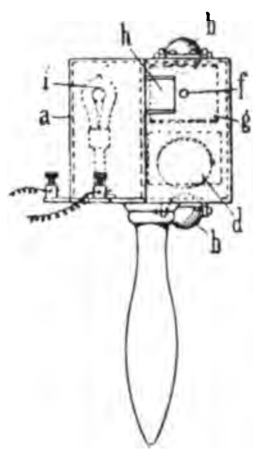


FIG. 223.

The apparatus is placed with the two spheres *b* against the forehead of the patient. The eye *c* of the latter looks into the concave mirror *d*; the observer (whose eye is represented by *e*) looks through the aperture *f* on to the plane mirror *g* and sees the image of the background of *c* formed by the mirror *d*. At the same time the eye *e* looks directly on to the pupil of *c* in the direction of the dotted line in Fig. 222, and can direct a mark provided in the instrument on to the centre of the pupil. A totally reflecting prism *h* is provided near the aperture *f* and reflects the rays from a small electric lamp *i* into the patient's eye. The prism *h* and the source of light are concealed from the observer looking through the aperture *f*.*

* *Reference* Lionel Laurence. "Visual Optics and Sight Testing" published by author, 1912.—Trans.

CHAPTER XVI.

Aplanatism.

159. Cartesian Systems.

As already explained in § 49, those conjugate points that are free from aberration and for which the sine condition is fulfilled, are called **Aplanatic points.** Assuming the object and image to be real, a system of centred spherical surfaces cannot produce true aplanatism ; such a system is able to satisfy the conditions for one zone only, or in the most favourable case, for several separated zones. A centred optical system which produces aplanatic imagery for all zones over a finite aperture is called **Aplanatic.** The author has shown* that an aplanatic system must possess at least two surfaces that are not spherical. The first condition for aplanatism, viz., the aberrationless image formation for all zones, has been a subject of interest to mathematicians for many years ; Descartes in particular has dealt with this problem, and on that account aberrationless systems will be referred to below as **Cartesian Systems.** Systems of this kind have also been called aplanatic, but in view of the conditions for aplanatism defined by Abbe, this term should be carefully avoided.

The earliest treatment of aberrationless systems is to be found in a work by Descartes,† which appeared in 1637. In Chapter 8 of this work, Descartes deals with the question of deducing that form of lens which produces images free from aberrations.‡ He determines the surface which furnishes aberrationless image formation for a given position of a conjugate pair of real points. As will be seen later, this surface is generally of the fourth order. In certain cases, if one of the aberrationless points is situated at infinity for example, the surface of the fourth order degenerates to a conic section (hyperboloid or ellipsoid).§

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* *See* "Notizen über aplanatische Abbildung" in Der Mechaniker ; Vol. 18, Pts. 15 and 16.

† Descartes "Discours sur la methode, pour bien conduire la raison et chercher la verité dans les sciences. Plus la dioptrique, les météores et la geometrie, qui sont les essais de cette méthode."

‡ *See* also an abstract of the treatment of this question in "Die Geschichte des Fernrohres bis auf die neueste Zeit" by Servus ; Berlin, 1886, p. 49 *et seq.*

§ For a treatment of the forms of the resulting lenses, the reader is referred to the works of Joseph Petri and W. Pscheidel in "Physikalische Zeitschrift, 1905, pp. 511 and 632. *See* also von Rohr "Bilderzeugung in optischen Instrumenten," Berlin, 1904, p. 338 ; Grusinzew, "Strahlenbrechung in Medien, welche von irgend welcher Oberfläche begrenzt sind" (Proceedings of the Kharkov Mathematical Society, 1889) : and the patent specifications Nos. 208,030 and 216,194 by Meyer of Saarbrücken."

In addition to such lenses in which spherical aberration has been entirely eliminated, it has also been sought by means of additional zonal polishing to improve spherical lenses even of the large sizes required for astronomical purposes. Such surfaces, deviating by only a *small amount* from the spherical, are best described as *deformed or figured surfaces*. The deformation or figuring of surfaces in optical systems in which the free lens surfaces themselves form the entrance apertures, and therefore in which all bundles of rays, central and oblique, must pass through the same parts of the lens surfaces, can influence only the spherical aberration of axial points. On the other hand, in optical systems such as eye-pieces and wide-angle photographic objectives, in which the entrance and exit apertures of the system are situated at greater or smaller distances from one or several of the surfaces, other errors of image formation, particularly for bundles of great obliquity, can be influenced by systematic deformation.* According to the patent specification mentioned in the footnote, the aberrations at the intersection of oblique bundles (magnification errors, curvature of image surface, astigmatism, coma) which are present in all truly spherical lens surfaces, are eliminated, at least partially, by systematic deformation.

Images produced by Cartesian systems, in spite of freedom from aberration, are subject to considerable defects. If for the aberrationless pair of points the sine condition is also fulfilled, the image formation is aplanatic and a surface element on the optical axis and perpendicular to it is imaged over its whole extent without spherical aberration. There is no doubt that aplanatic systems will play an important part in the future of technical optics, as soon as ways and means have been found which will enable surfaces, differing more or less from the spherical, to be ground with the same accuracy as spherical surfaces. A dissertation by Martin Linneman of Sehlde, Göttingen, 1905, deals with the mathematical theory of aplanatic lenses of this kind and gives the constructional data of an objective of aperture ratio $f/2$ which is truly aplanatic. The available field of view of the objective is also derived by trigonometrical calculation of the ray paths. In addition, the paper contains general formulæ for the trigonometrical computation of deformed lens systems and the question of the possibility of achromatising such lenses is investigated. It may be remarked that for the purpose of

* A detailed explanation will be found in the specification of German Patent No. 119,915 by C. Zeiss, Jena.

the achromatisation of aplanatic systems, modern optics provides to a certain extent the means of constructing such lenses by combining glass of equal refractive indices but different dispersions; so that the aplanatic condition is not altered, whilst the elimination of chromatism can be effected.

160. Optical Length.

Considering light as a wave disturbance, a wave-surface from a given luminous point is a surface of equal phase, *i.e.*, a surface containing all those elements in space which, in consequence of the light disturbance, have moved by equal amounts from their positions of rest. According to the wave theory the light proceeding from a luminous point or surface element is, after the lapse of a certain period of time, extended over a wave surface, the rays of light being normal to this surface (law of Malus). In a homogeneous medium such as air, glass, fluids, in which the disturbance proceeds outwards with the same speed in all directions, the wave-surface is a sphere with the luminous point as centre. If the disturbance is propagated in a medium with a velocity v (metres per sec.) and if v_0 be the velocity in free space, the refractive index of the medium is (§1)

$$n=\frac{v_0}{v}$$

If the light enter another medium of refractive index n', separated from the first by a refracting surface, and proceeds in the new medium with a velocity v' then

$$n'=\frac{v_0}{v'}$$

hence

$$\frac{n'}{n}=\frac{v}{v'}$$

The refractive indices of the two media are inversely proportional to the velocities of light in those media.

In the second medium, a new wave-surface is formed and the rays of light in this medium are normals to this surface. In general, the surface is not spherical. The same holds when light is refracted in turn at a number of surfaces.

Fig. 224 shows a system of any three surfaces. Of the infinite number of rays proceeding outwards from P, only three are shown, which are incident on the first surface at A, A_1, A_2, on the second at B, B_1, B_2 and on the third at

C, C_1, C_2. On the wave-surface in the last medium are three points D, D_1, D_2 such that the rays CD, C_1D_1, C_2D_2 in this medium are normals to this surface. Whether the light disturbance proceeds along the first, second or third ray, it must reach the points D, D_1, D_2, at the *same* time.

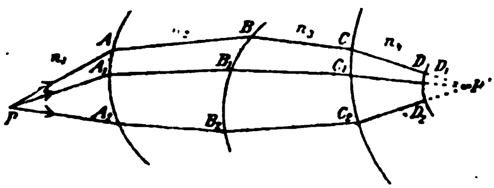

FIG. 224.

It is required to find an expression for this time T.

There are four media concerned, of refractive indices n_1 to n_4 and light velocities v_1 to v_4. Considering the first ray, let

$$PA = l_1 \qquad AB = l_2 \qquad BC = l_3 \qquad CD = l_4$$

Then the time T for the light to travel from P to D is

$$T = \frac{l_1}{v_1} + \frac{l_2}{v_2} + \frac{l_3}{v_3} + \frac{l_4}{v_4}$$

and since

$$\text{refractive index} = \frac{\text{velocity in free space}}{\text{velocity in the medium considered}}$$

we obtain

$$T = \frac{1}{v_o}(n_1 l_1 + n_2 l_2 + n_3 l_3 + n_4 l_4) = \frac{1}{v_o}\Sigma nl \qquad (1).$$

Whatever path the light from P is considered to traverse, after the time interval T the disturbances are in the same phase at the points D, etc., *i.e.*, on the wave-front in the last medium. This surface, therefore, is determined by the condition that the time T is constant for all points on it.

Since v_o is constant, we may write as the equation to the surface,

$$\Sigma\,(n\,.\,l) = constant \quad \ldots \qquad \ldots \qquad (2).$$

The length of the path multiplied by the refractive index of the medium is called the **Optical length.** The wave-front may therefore be defined as the surface of constant optical length.

161. Condition for Cartesian Image-Formation.

If the refracting surfaces are so chosen that the wave-surface determined by the points D, D_1, D_2 is a spherical surface, then all rays, since they are normals to this surface, intersect at the point P', the centre of the sphere (Fig. 224); *i.e.*, P' is the aberrationless image of P. When the light arrives at P', the wave-surface shrinks to a point. The distances l_4 in equation (2) are now to be reckoned to P' and between P and P' the relation holds, viz. :—

$$\Sigma\ (n\,.\,l) = constant \quad \dots \quad \dots \quad (3).$$

A pair of points is free from aberration when the optical lengths between them are constant for all paths of the rays. Evidently the results obtained are quite general and are not limited to three refracting surfaces. Equation (3) expresses the essential condition for aplanatic image formation, but it is not adequate. In calculating aplanatic systems, the condition that the refraction at the individual surfaces must follow the law of refraction must be taken into account. For any surface, the law of refraction may be expressed analytically as a differential equation. To obtain freedom from aberration for several successive surfaces, the resulting series of differential equations whose variables are the co-ordinates of the surfaces, has a particular solution in equation (3).*

162. Cartesian Surfaces.

In Fig. 225 let $L\,L_1$ be the trace, in the plane of the paper, of a Cartesian surface whose vertex is S. $P\,A\,P'$ is any ray from P to its aberrationless image point P'.

$$\text{Let } A\,P = l \qquad\qquad A\,P' = l'$$

then from (3)

$$n\,l + n'\,l' = n\,l_o + n'\,l_o' \quad \dots \quad (4).$$

is the condition for freedom from aberration, where $P\,S = l_o$ and $P'\,S = l_o'$ and n and n' are respectively the refractive indices of the media in front of and behind the surface. Dropping the perpendicular $A\,B = y$ and putting $S\,B = x$, then x and y are the co-ordinates of the point A on the surface referred to rectangular axes with S as origin, and

$$l = \sqrt{y^2 + (l_o + x)^2} \qquad\qquad l' = \sqrt{y^2 + (l'_o - x)^2}$$

Substituting these values in equation (4), we obtain the equation of the Cartesian surface ; it is of the fourth order.

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* *See* Der Mechaniker, Vol. 18, No. 15.

163. Cartesian Lenses.

In Fig. 225 let the medium in front of the surface be air ($n = 1$) and that behind the surface, say, glass. A sphere may be described about P' as centre and with S_1 as vertex, this sphere forming a second boundary between the glass and the air behind it. Since this sphere is penetrated normally by the refracted rays, it does not disturb the condition of freedom from aberration, and a lens *free from aberration*, or a **Cartesian lens,** is formed.

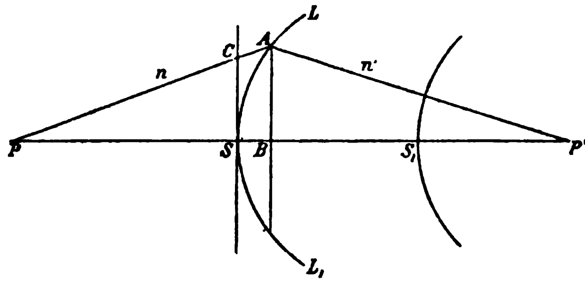

FIG. 225.

A lens of this kind, however, can never be aplanatic, since the points of intersection of incident and refracted rays all lie on the Cartesian surface, whereas, to obtain aplanatic image-formation they must lie on a sphere as pointed out in § 50.

It is possible, however, to form aplanatic systems by combining Cartesian lenses. For example, consider an object plane to be imaged to the right and left by two equally distant and congruent Cartesian lenses; two images of equal size which are conjugate to one another, and moreover aplanatic, are thus obtained, since each ray which proceeds from the axial point of one image intersects the axial point of the other image at the same inclination to the axis; consequently the sine condition is fulfilled. The resultant effect, of course, is merely a projection of the image from one place to another.

164. Special Cases of Cartesian Lenses.

If a plane be considered perpendicular to the axis through the vertex of the Cartesian surface, cutting the incident ray in C, then the optical length of the incident ray may be divided into two portions, $n.PC$ and $n.CA$. If P be situated at a great distance, the portion of the ray path $n.PC$. may be assumed to be constant and the line CA becomes parallel to

the axis. The condition for Cartesian image-formation may then be simply stated; the optical length of the path from C to P' must be constant. In Fig. 226 the luminous point P is very distant, so that the rays are incident parallel to the axis. Let $CA = x$, $AB = y$ and assume that air lies to the left of the surface and glass of refractive index n to the right. Then we have as the condition for freedom from aberration

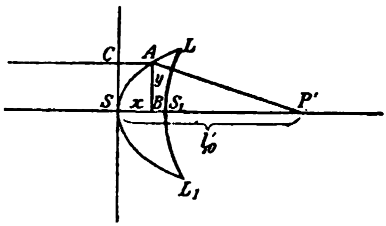

FIG. 226.

$$x + n \cdot AP' = l_0' \cdot n \,. \quad \ldots \quad (5)$$

where

$$AP' = \sqrt{y^2 + (l_0' - x)^2}.$$

From these equations

$$y^2 = 2\, l_0' x \left(1 - \frac{1}{n}\right) - x^2 \left(1 - \frac{1}{n^2}\right) \ldots \quad (6).$$

which is the equation to an ellipse referred to the vertex.

If b is the minor and a the major axis of the ellipse and e the eccentricity, then

$$\frac{b^2}{a} = l_0' \left(1 - \frac{1}{n}\right),$$

$$\frac{b^2}{a^2} = 1 - \frac{1}{n^2}$$

Further, since

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

$$e = \frac{1}{n}.$$

The eccentricity of the ellipse is equal to the reciprocal of the refractive index.

Further

$$a = \frac{l_0'}{1 + \frac{1}{n}}$$

It will be seen immediately from the last equation, that $l_0' = SP'$ is equal to the distance from the vertex S to the second focus of the ellipse, so that the image P' coincides with the latter point.

If now we limit the medium of refractive index n (glass) by a sphere with centre P' and radius $P'S_1$ we obtain an aberrationless lens, since the sphere is penetrated by normal rays, the paths of the latter consequently not being disturbed. This lens is shown in Fig. 227.

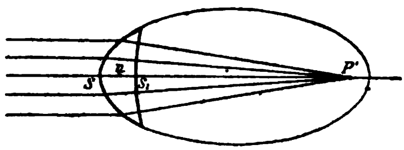

FIG. 227.

Suppose now that the medium in the object space be glass of refractive index n, whilst that of the image space be air. Assuming that the incident rays are parallel to the axis, then the condition corresponding to equation (4) *i.e.*, for freedom from aberration of a surface and the formation of a real image point, (the surface must therefore present its concave side to the incident light), becomes,

$$AP' - nx = l_o'$$

where

$$AP' = \sqrt{y^2 + (l_o' + x)^2}$$

From these two equations,

$$y^2 = 2\, l_o x\,(n - 1) + x^2\,(n^2 - 1)$$

which is the equation to a hyperbola referred to the vertex.

The eccentricity of the hyperbola is found to be equal to the refractive index and the point P' lies at the focus of the other branch of the hyperbola. In order to obtain an aberrationless lens the medium of refractive index n must be enclosed by means of a plane surface perpendicular to the axis. This lens with vertices S and S_1 and the corresponding hyperbola are shown in Fig. 228, P' being the aberrationless focal point.

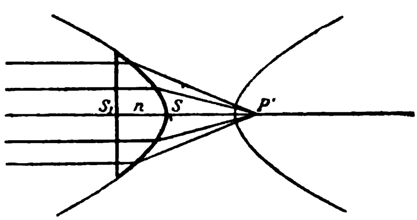

FIG. 228.

165. Aplanatic Inverting Systems.

Lenses of the form shown in Figs. 227 and 228 may be used as aplanatic inverting systems with a magnification of unity, by so placing two such lenses that the rays between them are parallel. The lenses of Fig. 227 must be placed

with their convex sides towards one another ; those in Fig. 228 with their plane sides towards one another. The latter combination may obviously be drawn together into a single lens, bounded on both sides by congruent hyperboloids. In both cases on account of the symmetry of ray paths, the sine condition is fulfilled (*see* § 163).

166. Aplanatic Points of Spherical Surfaces. Aplanatic Menisci of the First Type.

A spherical surface, which separates two media of refractive indices n and n', has the important property that aplanatic image-formation takes place for a certain definite pair of conjugate points. This property can be demonstrated most simply by Young's graphic construction for determining the position of a refracted ray, from which Weierstrass has derived his method, Fig. 229.

Let the centre of the spherical surface be M. With centre M draw two concentric circles of radii $r.\frac{n'}{n}$ and $r.\frac{n}{n'}$ where r is the radius of the refracting spherical surface. Assume the plane of the paper to be a principal section of the surface. A ray of light incident in the direction of the arrow meets the refracting surface at A, and on being produced cuts the circle of radius $r.\frac{n'}{n}$ in P. The line joining MP cuts the circle of radius $r.\frac{n}{n'}$ in P'. Then AP' is the refracted ray. In Fig. 229 it is assumed that $n' > n$. Join AM. Then the triangles AMP' and AMP are similar, since they have an angle in each equal and the sides about that angle proportional.

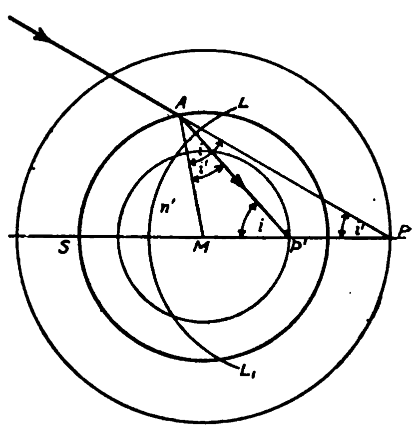

FIG. 229.

The angle of incidence is $P\hat{A}M = i$.

The angle of refraction is $P'\hat{A}M = i'$.

From the construction $A\hat{P'}M = i$ and $A\hat{P}M = i'$.

If this construction be repeated, it will be seen that all rays directed towards P pass through P' after refraction. In

other words the point P is imaged without aberration at the point P'. Further, the sine condition is fulfilled; for assuming the straight line $MP'P$ to be the optical axis, the incident and refracted rays make with it the angles i'' and i, and from the law of refraction,

$$\frac{\sin i''}{\sin i} = \frac{n}{n'} \quad \dots \quad \dots \quad (1).$$

Hence the sine condition is fulfilled.

From Fig. 229

$$MP = r \cdot \frac{n'}{n} \qquad MP' = r \cdot \frac{n}{n'}$$

S is the vertex of the refracting surface, and

$$SP = r + r \cdot \frac{n'}{n} \qquad SP' = r + r \cdot \frac{n}{n'}$$

The quantity SP, according to the sine convention, is to be reckoned negative (the object point lying to the right of the refracting surface). Let the conjugate intercepts of the aplanatic points be represented by s and s'; then

$$s = -r \cdot \frac{n + n'}{n} \quad \dots \quad \dots \quad (2)$$

$$s' = r \cdot \frac{n + n'}{n'} \dots \quad \dots \quad \dots \quad (3).$$

Four particular cases may be distinguished:

1. *r positive*; the refracting surface presenting its convex side to the incident light. If the first medium be air and the second glass of refractive index n, then

$$s = -r\,(n + 1)$$

$$s' = r\,\frac{(n + 1)}{n}.$$

This case corresponds to the path of the rays shown in Fig. 229. If now a sphere with suitable radius and P' as centre be described, cutting the principal section of the refracting spherical surface in the points L and L_1 and crossing the axis normally, then LSL_1 is an **aplanatic meniscus**, since the second spherical surface is penetrated normally by the refracted rays, and consequently the paths of the latter are not altered. This meniscus renders more convergent the incident light which is converging on a virtual object point P.

2. *r positive;* the first medium being glass of refractive index n and the second one air. Putting $n' = 1$ in equations (2) and (3) we obtain :

$$s = -r \cdot \frac{n+1}{n} = PS$$

$$s' = r(n+1) = P'S$$

Fig. 230 shows the aplanatic points for $n = 1{\cdot}5$. M is the centre and S the vertex of the refracting surface. If a sphere be described around P as centre and with vertex S_1 thus separating the glass from air on the left, a **negative aplanatic meniscus** with virtual object point P is formed.

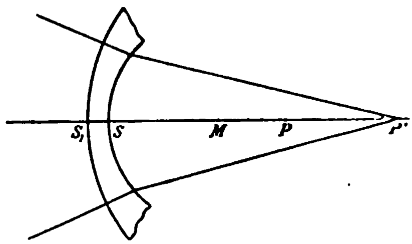

FIG. 230.

3. *r negative;* refractive indices as in case 2, so that the incident ray is in glass. Putting $-r$ for r and $n' = 1$ in equations (2) and (3), we obtain :

$$s = r \cdot \frac{n+1}{n}$$

$$s' = -r \cdot (n+1).$$

Since s is positive, the object point is *real* and the image point *virtual.*

In Fig. 231, M is the centre of the refracting surface; $MS = r$. The aplanatic points are P and P'. If, with P as centre, a sphere be described whose vertex is S_1, separating the glass from the air in front of it, we obtain a **positive aplanatic meniscus** with *real object point.* The divergence of the light from P is diminished.

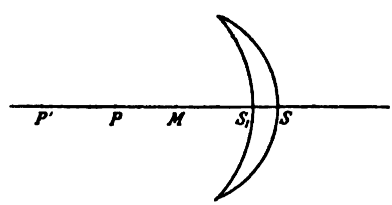

FIG. 231.

4. *r negative;* the concave side of the refracting surface towards the light; let the incident ray be in air. Putting $-r$ for r, $n = 1$, $n' = n$ in equations (2) and (3), we obtain :

$$s = r \cdot (n+1)$$

$$s' = -r \cdot \frac{n+1}{n}.$$

If the glass be bounded behind by a spherical surface whose centre is P' and vertex S_1 we obtain a **negative aplanatic meniscus.** Fig. 232 shows the positions of the aplanatic points. It will be seen that the divergence of the light coming from P is increased.

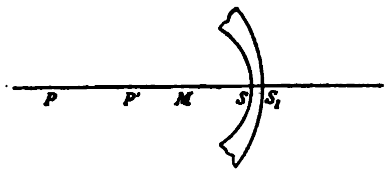

FIG. 232.

The four forms of aplanatic menisci thus obtained produce at the aplanatic image points, images which are n times as large as the object.

The most important requirement in optics, *i.e.*, the formation of a real image of a real object, cannot be realised by their help. The use of aplanatic menisci is therefore limited to field-lenses, parts of illuminating systems, etc. Most frequently use is made of the third form in which a diminution of the divergence of the incident light takes place; an effect which obtains in the case of immersion systems of microscope objectives on account of the hemispherical form of the front lens.

167. Aplanatic Menisci of the Second Type.

The aplanatic menisci dealt with in § 166 are based on the principle that the bundle of rays originating at one aplanatic point of a spherical surface as object, and forming after refraction at the surface an image at the second aplanatic point, then intersects a new spherical surface on the object side or image side, in such a way that all the rays penetrate it normally, the paths of the rays thereby not being altered. The aplanatic bundle of rays may, however, cut a new spherical surface in another manner without interfering with the aplanatism. In this case it is so arranged that the rays are directed towards the aplanatic point of the new spherical surface. Thus a *second* aplanatic refraction is produced. Proceeding on these lines, we obtain a number of aplanatic menisci, which will be termed **aplanatic menisci of the second type,** the character of which will now be investigated.

Let L (Fig. 233) be a system which produces by some means an aplanatic image at A_0, the light being incident in the direction of the arrow.

In the path of the rays, after passing through L, is placed an aplanatic lens L_1 with vertices S_1 and S_2 and radii

$$\left.\begin{aligned} r_1 &= \frac{-a}{n+1} \\ -r_2 &= \frac{-a-nd}{n+1} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2)$$

where $a = -S_1A_0$ and the thickness $d = S_1S_2$ is suitably chosen.

The image point A_0 will be displaced to A_2, and for the determination of the position of A_2 we have the relation

$$S_2A_2 = -a - nd \quad \ldots \quad \ldots \quad (3).$$

The new image point A_2 has the property that the ray which, at a finite inclination and fulfilling the sine condition,

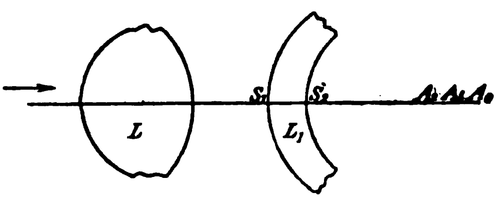

FIG. 233.

passes exactly through A_0, is also directed, after refraction through the system $L + L_1$, exactly through the point A_2; *i.e.*, without aberration and at the same time fulfilling the sine condition. If only the spherical aberration is eliminated for A_0 and the sine condition is not satisfied, then the same will be true for the point A_2.

168. Semi-Aplanatic Lenses of Zeiss.

In the case of the Linneman lenses (§ 159) both surfaces deviate from the spherical form. Lenses have been introduced by Zeiss,* however, in which only one surface is non-spherical, the other being spherical. Systems of this kind are not truly aplanatic, but either the sine condition is fulfilled for the outer zones and spherical aberration completely corrected; or, the sine condition is entirely satisfied with spherical aberration eliminated for one zone only. On this account they may be termed **Semi-aplanatic Systems.**

* *See* French Patent No. 401,974 and British Patent No. 7,144 of 1908.

Figs. 234–239 represent meridional sections of six lenses of this kind. In each case the quantities involved are as follows:

- r radius of non-spherical surface near the vertex.
- R radius of spherical surface.
- u half aperture angle of the beam corresponding to the spherical surface.
- v half the aperture angle of the beam corresponding to the non-spherical surface.
- l the distance of the lens from the intersection point of the beam for the aperture angle $2u$.
- m the corresponding distance for $2v$.
- d the thickness of lens along axis.
- D the diameter of the lens.

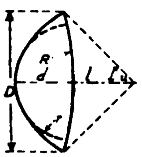

FIG. 234.

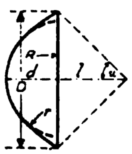

FIG. 235.

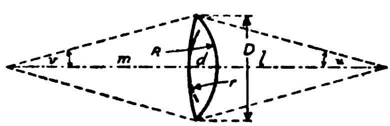

FIG. 236.

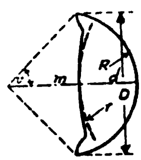

FIG. 237.

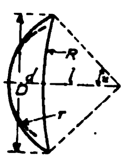

FIG. 238.

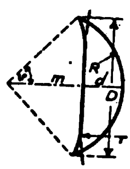

FIG. 239.

In the following table are given numerical values for the six cases represented by the figures, as well as eighteen other examples arranged in increasing values of refractive index n. The focal length f is assumed to be 100. The radii of convex surfaces are assumed positive, those for concave surfaces, negative. The semi-aperture angles v and u are taken as positive when the intersection point of the beam lies on the side of the non-spherical and the spherical surfaces respectively. This applies also to the signs of the corresponding distances m and l. In the examples 1 to 11, the algebraic sum of the semi-aperture angles v and u amounts to 30°; in the remaining examples to 45°. In the two chief groups, viz., 1 to 11 and 12 to 24, the examples follow one another in steps of 15° of v and u. Thus the same pair of v

and u occurs in each group twice; once (Nos. 1–5 and 12–17) in conjunction with $n = 1{\cdot}5$ and again (Nos. 7–11 and 10–24) with refractive index $n = 1{\cdot}75$. In addition, two more examples are given (Nos. 6 and 18) in which $v = 0$ and the refractive index is 1·618. The table gives the calculation of a semi-aplanatic lens for the case in which the radius r is to be so deformed that spherical aberration is completely corrected, the values n, v, u, r, R, d, m and l being given. Two examples are included in which the refractive index is 1·618 and the half aperture angle $v = 0°$ because in the case of parallel rays on one side, which is perhaps the most important case in practice, this results in a plane surface instead of a spherical surface; and the radius r of the deformed surface at the vertex becomes independent of the half aperture angle u. If the refractive index is smaller than those of the most commonly used kinds of glass, then over a certain range both radii R and r are positive, *i.e.*, the surfaces are convex. This range extending between v slightly greater than 0 on the one hand, and u slightly greater than 0 on the other, includes the examples 2–4 and 13–16. If those pairs of v and u, within this range, in which v is considerably greater than u be excluded, leaving such examples as 2 and 3, and 13–15, it is seen that r is constantly smaller than R. On the other hand in the case of those pairs in which u takes the value 0, r is greater than R, as, for example, in numbers 4 and 16.

No.	n	v degrees.	u degrees.	r	R	d	m	l	Fig.
1	1·5	−15	45	36	−104	21	−167	44	—
2	1·5	0	30	55	448	27	∞	84	—
3	1·5	15	15	87	105	29	191	189	236
4	1·5	30	0	228	62	27	85	∞	—
5	1·5	45	−15	−137	38	21	45	−168	—
6	1·618	0	30	62	∞	22	∞	87	—
7	1·75	−15	45	40	− 71	14	−165	48	—
8	1·75	0	30	68	−605	18	∞	89	—
9	1·75	15	15	137	155	19	195	194	—
10	1·75	30	0	−1260	71	18	89	∞	—
11	1·75	45	−15	− 75	41	14	48	−165	—
12	1·5	−15	60	40	−131	42	−224	36	—
13	1·5	0	45	55	390	59	∞	64	234
14	1·5	15	30	66	138	68	277	117	—
15	1·5	30	15	80	96	67	128	265	—
16	1·5	45	0	121	72	59	73	∞	237
17	1·5	60	−15	−904	48	42	41	−233	—
18	1·618	0	45	62	∞	47	∞	71	235
19	1·75	−15	60	44	− 78	28	−219	43	—
20	1·75	0	45	68	−540	39	∞	75	238
21	1·75	15	30	98	253	45	286	132	—
22	1·75	30	15	165	121	45	136	281	—
23	1·75	45	0	2300	77	39	78	∞	239
24	1·75	60	−15	− 96	47	28	45	−222	—

169. Optical Properties of the Cardioid.

After Schwarzschild* had drawn up general conditions for the aplanatism of reflecting systems, Siedentopf explained a particularly interesting special case. He showed that for parallel incident rays, aplanatic image-formation occurs by reflection at a cardioid in conjunction with reflection at a spherical surface.† Before proving this property, we will first consider a corollary.

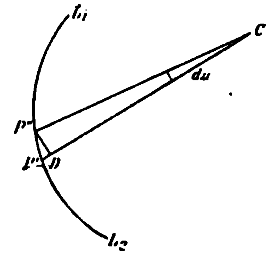

FIG. 240.

In Fig. 240, let L_1L_2 be a reflecting surface; CP' and CP'' are two rays very close together and incident on this surface. $P''D$ is the perpendicular on CP', and $P''P'$ is perpendicular to the normal at P'. Hence $D\hat{P''}P'$ must be equal to the angle of incidence i, of the ray CP' and we have

$$\tan i = \frac{P'D}{P''D}$$

Let the radius vector $P'C = R$, and the infinitely small angle $P'CP'' = du$; then $P''D = Rdu$, whilst $P'D$, *i.e.*, the small decrement of R, is represented by $-dR$,

then

$$\tan i = -\frac{dR}{R \, . \, du} \quad \ldots \quad \ldots \quad (1).$$

Let Fig. 241 represent a cardioid on which are situated the points C and P'. Let the point C on the axis ZZ be the origin of co-ordinates.

Then the polar equation of the cardioid is

$$R = r\,(1 + \cos u) \quad \ldots \quad \ldots \quad (2).$$

Let M be a point distant $CM = \frac{r}{2}$ from C and with centre M describe a circle of radius r. It is required to prove that a ray at any height h, incident in the direction OP and reflected at P towards P' is so reflected at the cardioid that it passes through the point C.

* Proc. Imp. Scientific Soc., Göttingen; Math.-Phys. Section; New Issue; Vol. IV. 1905, No. 2, p. 23.

† *See* "Ueber einen neuen Fortschritt in der Ultramikroskopie," Trans. German Phys. Soc., XII year, No. 1. 1910.

By differentiating (2) we obtain

$$dR = -r \sin u.\, du.$$

Hence

$$\tan i = \frac{r}{R} \sin u$$

$$= \frac{\sin u}{1 + \cos u}$$

$$= \tan \frac{u}{2}$$

or

$$i = \frac{u}{2}.$$

If now we assume that a ray emanating from C and reflected from the cardioid at P' cut the axis ZZ at B', then the triangle $P'B'C$ is isosceles and we obtain immediately

$$\cos u = \frac{R}{2\,B'C}.$$

Further

$$B'M = B'C - MC = \frac{R}{2 \cos u} - \frac{r}{2}$$

$$= \frac{r\,(1 + \cos u)}{2 \cos u} - \frac{r}{2} = \frac{r}{2 \cos u} = B'M \qquad (3).$$

If we assume further, that the incident ray OP is so drawn that it is reflected to P', then it is required to prove that $P'P$ produced passes through the point B'. Let $P'P$ produced cut the axis ZZ in the point B. Draw the incidence normal $MP = r$. Then from the law of reflection, the triangle PBM is isosceles and

$$BM = \frac{r}{2 \cos u}$$

Hence from equation (3)

$$BM = B'M$$

and B and B' coincide.

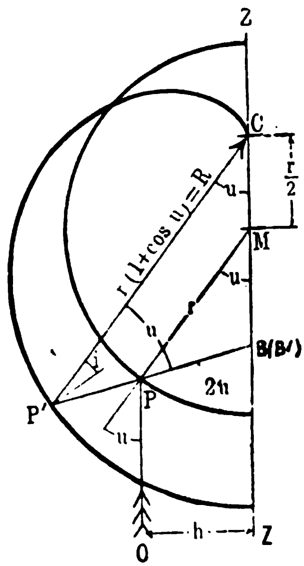

FIG. 241.

The combination of the two reflections gives then an aberrationless intersection of rays, *i.e.*,

the point C is the aberrationless focal point of parallel incident rays. Further from Fig. 241

$$\sin u = \frac{h}{r} \text{ or } \frac{h}{\sin u} = r = \textit{constant.}$$

where r is also the focal length of the combination of reflectors. Since the latter equation expresses the condition for the fulfilment of the sine condition, the imagery is *aplanatic.*

These properties of the cardioid form the basis of the so-called cardioid condenser of Zeiss.

170. Use of Aplanatic Systems.

The employment of aplanatic, semi-aplanatic or Cartesian systems might be adopted more widely than heretofore with the further development of the technique of glass working, and it is within the bounds of possibility that along these lines a revolution in technical optics lies before us. It may be remarked that the fulfilling of achromatism appears to offer no insurmountable difficulties. If, for example, aplanatic lenses made of varieties of glass of equal refractive indices but different dispersion are combined, the aplanatism is not disturbed, whilst the conditions for the fulfilling of achromatism are not restricted, as we have already explained.

We will finally refer to some recent applications of these principles.

In searchlight systems of two separated convex lenses, a greater spherical correction is produced by the use of deformed surfaces.*

In complex condensers for projection purposes deformed surfaces are likewise used.†

Aplanatic field-lenses consist of three separated single lenses.‡

Besides the cases named above, deformed surfaces have recently been used also in the correction of the path of the rays in spectacle lenses (*see* § 150).

* *See* German Patents Nos. 341,677 and 353,870 by C. Zeiss.
† *See* German Patents Nos. 341,673 and 341,674 by C. Zeiss.
‡ German Patent No. 316,900.

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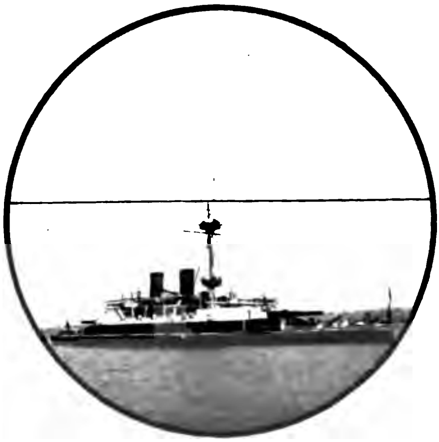

Fig. 1.

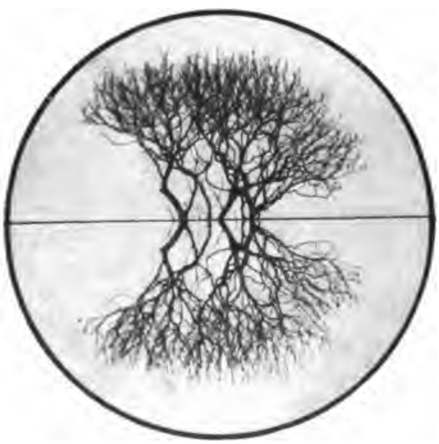

Fig. 2.

Plate 1.

BIBLIOGRAPHY.

ABBE, E. Abbe's important work will be found fully dealt with in Czapski "Die Theorie der optischen Instrumente nach Abbe"; also *see* references to his original papers in Southall "Principles and Methods of Geometrical Optics."

BECK, A. Triple Mirrors. *Zeitschr. für Instrumentenkunde.* **1887.** P. 380.

BECK, CONRAD. The Theory of the Microscope. *Journal Society of Arts, Cantor Lecture.* **1908.**

BOUASSE, H. Traité d'Optique Géométrique supérieure, **1918.**

CHESHIRE, F. J. The Modern Rangefinder. **1917.**

The Stereoscopic Rangefinder; *Optician.* **1917.**

True-to-Nature Photography, *Optician*; Vol. XLV. No. 1,160. **1913.**

CHESHIRE, R. W., and SMITH, T. Constructional Data of Small Telescope Objectives. **1917.**

COLE, R. S. Treatise on Photographic Optics. **1899.**

CZAPSKI, S. Theorie der optischen Instrumente nach Abbe. Breslau. **1893.**

Ueber neue Arten von Fernrohren insbesondere zum Handgebrauch. *Gesell. für Mechanik und Optik.* **1895.**

DALLMEYER, T. R. Telephotography. **1899.**

DRUDE, P. Theory of Optics. Trans. by Mann and Millikan. Longmans. **1902.**

DRYSDALE, C. V. Small Telescopes and Binoculars. *Optical Convention.* **1905.**

EDSER, E. Light for Students. Macmillan. **1915.**

EQUEVILLEZ, R. D'. Untersee—und Tauchboote. Kiel. **1905.**

FERRARIS, G. *See* Lippich. Dioptric Instruments.

FISCHER. Ueber Ferienkurse für Wissenschaftliche Mikroskopie, *Zeitschr. für Wiss. Mikrosk.* Vol. 27.

GAGET, M. La navigation sous-marine. Paris. **1901.**

GLAZEBROOK, R. T. Glass for Optical Purposes. *Journ. Society of Arts.* **1902.**

GLEICHEN, A. Lehrbuch der geometrischen Optik. Leipzig. 1902.

Einführung in die medizinische Optik. Leipzig. 1904.

Vorlesungen ueber photographische Optik. Leipzig. 1905.

Leitfaden der praktischen Optik. 1906.

Ueber die Bedeutung der Dioptrie und Konvergenz-rechnung. 1909.

Notizen ueber aplanatische Abbildung. *Der Mechaniker* (Vol. 18).

Die Grundgesetze der naturgetreuen photographische Abbildung. 1910.

Galilean Telescope. *Optician.* 1917.

GLEICHEN, A., and KLEIN, E. Schule der Optik. Leipzig. 1914.

GRUSINZEW. Strahlenbrechung in Medien, welche von irgend welcher oberfläche begrenzt sind. *Kharkov. Math. Soc.* 1889.

GULLSTRAND, A. Ueber die Bedeutung der Dioptrie. *Archiv für Ophthalmologie.* 1899.

HARTING, H. Optischen Hilfsbuch für Photographierende. Berlin. 1909.

Ueber die Theorie des Apochromatkollineares. *Photo. Korresp-* 1901.

Das Heliar. *Photo. Korresp.* 1902.

HEATH, R. S. Treatise on Geometrical Optics. 2nd Ed. Cambridge. 1895.

HELMHOLTZ, H. Handbuch der physiologische Optik. 3rd Ed. Hamburg. 1909.

HERMAN, R. A. Treatise on Geometrical Optics. 1900.

HINRICHS. Beitrag zur Theorie der natürlichen Blender Optischen Instrumente. Rostock. 1910.

HOFE, VON. Fernoptik. 1911.

HOVESTADT, H. Jena Glass and its Applications. Trans. by J. D. and A. Everett. London. 1902.

KÖHLER, A. Mikrophotographische Untersuchungen mit ultra-violettem Licht. *Zeitschr. für Wiss. Mikrosk.* Vol. 21. 1904.

LAURANCE, LIONEL. Visual Optics and Sight Testing. 2nd Ed. London. 1920.

LEATHEM, J. G. Elementary Theory of the Symmetrical Optical Instrument. 1908.

LIPPICH, F. Dioptric Instruments. Trans. by O. Faber. 1919.

LUMMER, O. Contributions to Photographic Optics. Trans. by Silvanus P. Thompson. London. 1900.

MARRE, DE. Les instruments pour la mésure des distances. 1880.

MAXWELL, J. CLERK. On the elementary theory of Optical Instruments. *Phil. Mag.* 1856.

On the general laws of Optical Instruments. *Quart. Journ. Math.* 1858.

NICOLASSEE. Über die Bedeutung der Dioptrie-und Konvergenzrechnung. Berlin. 1909.

NIESIOLOWSKI-GAMIN. Ausgewählte Kapitel der Technik. Vienna.

NUTTING, P. G. Outlines of Applied Optics. 1912.

PERCIVAL, A. S. Geometrical Optics. 1913.

PELLETAN, A. Optique Appliquée. 1910.

POUILLET, MÜLLER. Lehrbuch der Physik. 1897.

PRESTON, E. Theory of Light. 1912.

PULFRICH, C. The Stereoscope. *Ency. Brit.* 11th Edition.

Neue stereoskopische Methoden und Apparate. 1903–1909.

ROHR, M. VON. Geometrical Investigation of the Formation of Images in Optical Instruments. Trans. by R. Kanthack. H.M. Stationery Office, London, 1920.

Zur Theorie der Fernrohrbrille. *Archiv für Ophthalmologie*. (Vol. 75).

Die Binokularen Instrumente.

Theorie und Geschichte des photographischen Objectivs. 1899.

ROSENHAIN, W. Glass Manufacture. 2nd Ed. 1919.

Optical Glass. *Journ. Society of Arts.* 1915.

RUDOLPH, P., AND WANDERSLEB, E. Ueber die Verzeichnungsfehler photographischen Objectivs. *Zeitschr. für Inst.* 1907.

SCHUSTER, A. Theory of Optics. 2nd Ed. 1919.

SERVUS. Die Geschichte des Fernrohres bis auf die neueste Zeit. 1886.

SIEDENTOPF, H. Die Vorgeschichte der Spiegelkondensoren. *Zeitschr. für wiss. Mikrosk.* 1908.

Ueber einen neuen Fortschritt in der Ultramikroskopie. *Trans. Germ. Phys. Soc.* 1910.

SMITH, T., AND CHESHIRE, R. W. *See* Cheshire, R. W.

SOUTHALL, J. P. C. The Principles and Methods of Geometrical Optics. New York. 1913.

Mirrors, Prisms and Lenses. New York. 1918.

STEINHEIL, A., AND VOIT, E. Applied Optics. Trans. by J. W. French. London. 1918.

STOLZE, VON. Die Stereoskopie und das Stereoskop. *Zeit. für wiss. Photographie.* Vol. v. 1907.

Ueber die bis jetzt mit dem Komparator auf astronomischen gebiete erhaltenen Versuchsergebnisse Astron. Wissenschaft.

STURM, J. C. Mémoire sur l'Optique. *Liouville. Journ. de Math.* 1838.

TAYLOR, H. DENNIS. System of Applied Optics. 1906.

TAYLOR, J. TRAILL. Optics of Photography and Photographic Lenses. 1904.

THOMPSON, SYLVANUS, P. *See* Lummer.

Optical Tables. 2nd Ed. 1907.

THORNER. Die Theorie des Augenspiegels und die Photographie des Augenhintergrundes. Berlin. 1903.

WANDERSLEB, E. *See* Rudolph, P.

WHITAKER, E. T. The Theory of Optical Instruments. 1907.

WOOD, R. W. Physical Optics. 1914.

ZANDER, A. Der Augenspiegel. Leipzig. 1859.

ZSCHOKKE, W. Anschauliche Darstellung der Entstehung und Hebung der Bildfehler. *Deutsche Mechanikerzeitung.* 1910.

Homogenität des optischen Glases. *Zeitschr. für Inst.* 1910.

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